M.A. (ECONOMICS) PART-I (Sem-I) PAPER-III

(BASIC QUANTITATIVE METHODS)

Section - A

Department of Distance Education
Punjabi University, Patiala

SECTION-A

Lesson No.

M.A. (ECONOMICS) PART–I **PAPER–III**
(BASIC QUANTITATIVE METHODS)

LESSON NO. 1.1 **AUTHOR : DR. VIPLA CHOPRA**

DIFFERENTIATION (SIMPLE FUNCTIONS)

Structure

1.1.0 Introduction :

Differential calculus is that branch of mathematics which is concerned with determining the rate of change of a given function due to a unit change in the independent variable. In other words it is the study of the changes that occur in one quantity when other quantities on which it depends change. In Economics and Business, we frequently come across such situations where we are interested in knowing:

(i) The change in total cost of production due to unit change in volume of output.

(ii) The change in demand for commodity due to unit change in the price of

commodity etc.

All these above problems are studied with the help of mathematical technique called differentiation. Before taking up the detailed study of derivatives, it becomes necessary first to define certain terms which are essential for the study of it.

Function of a Variable:

When two variables X and Y are connected in such a way that corresponding to each value of the first variable 'x' there is unique value of the second variable 'y' then second variable 'y' is said to be function of first variable x and function y is denoted by

$$Y = f(X)$$

The variable x is called independent variable and y is called dependent variable.

Limits of the Function:

In this section before defining limit we will study the behaviour of a function f (x) of a variable x, as x approaches a particular value say 'a'. When we say x approaches 'a', we mean that x takes successive values that gets arbitrary close to 'a' but x never equals a.

Symbolically we write $x \to a$ or Lt. $x = a$

and read as limits of $x = a$.

Let $Y = \frac{1}{2x}$ is a single valued function of x.

i.e. To each value of x there corresponds one and only one value of y. How does this function behave as a sequence of values allotted to x according to some law ?

From the above function, we get

x	:	1	2	3	4	5	.....1000.....
y	:	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{6}$	$\frac{1}{8}$	$\frac{1}{10}$	..... $\frac{1}{2000}$ .....

To the x sequence, there corresponds a y sequence. Y has been constructed according to some rule. It is not a collection of some arbitrary numbers. The idea one gets is that as x becomes larger and larger, y or f (x) becomes smaller and smaller. As x tends to infinity, y tends to zero. We can never make it equal to zero by making x larger and larger, but we can make it close to zero. Thus, when a sequence of values is allotted to x according to some law, the corresponding sequence of values of f (x) determines the limit to which the function approaches.

Let f (x) be a function defined for all values of x close to c except possibly at point c. Then L is said to be the limiting value of *f(x)* as x approaches c, if the numerical difference between *f(x)* and L can be made as small as we like by making the positive

difference between x and c small enough.

In symbols $\underset{\Delta x \to c}{Lt} f(x) = L$

Alternatively: *f (x)* tends to a limit L as x tends to c, if for each given $\in > 0$ however, small, there exists a positive number δ (that depends upon $\in$). Such that

$|f(x) - L| < \in$

for all values of x for which

$0 < |x - c| < \delta$

The condition $|f(x) - L| < \in$ is the same as $L - \in < f(x) < L + \in$. Therefore, it is clear that limit exists if *f(x)* can be confined to a small interval $(L - \in, L + \in)$ by confining x to the interval $(c - \delta, L + \delta)$ which depends upon $(L - \in, L + \in)$

Theorems:

1. If a is constant $\underset{x \to c}{Lt}\, a = a$

2. If a and b are constants $\underset{x \to c}{Lt}(a \quad + b) = ac + b$

3. If $\underset{x \to a}{Lt} f(x) = p$ and $\underset{x \to a}{Lt} \phi(x) = q$, then

   (i) $\underset{x \to a}{Lt}[f(x) \pm \phi(x)] = \underset{x \to a}{Lt} f(x) \pm \underset{x \to a}{Lt} \phi(x) = p \pm q$

   (ii) $\underset{x \to a}{Lt}[f(x) \times \phi(x)] = \underset{x \to a}{Lt} f(x) \times \underset{x \to a}{Lt} \phi(x) = pq$

   (iii) $\underset{x \to a}{Lt} = \dfrac{f(x)}{\phi(x)} = \dfrac{\underset{x \to a}{Lt} f(x)}{\underset{x \to a}{Lt} \phi(x)} = \dfrac{p}{q}$

   (iv) $\underset{x \to a}{Lt}[cf(x)] = c \underset{x \to a}{Lt}[f(x)] = cp$, where c is constant

   (v) $\underset{x \to a}{Lt} \log f(x) = \log \underset{x \to a}{Lt} f(x) = \log p$, provided p > 0

1.1.1 Objectives:

After reading this lesson, you will be able to

* define the concept of derivatives.
* know the geometrical interpretation of derivatives.
* learn about different methods of obtaining derivatives.

1.1.2 Derivatives:

The process of obtaining a derivative is called the differentiation of the function. Let y = f (x), since y depends upon x any change in x results in change in y. Suppose Δx represents change in x and Δy represents change in y.

The ratio $\frac{\Delta y}{\Delta x}$ is called the incremental ratio. Now, if the change in the independent variable x is very small i.e. $\Delta x \to 0$ then the ratio $\frac{\Delta y}{\Delta x}$ may approach to a definite limited value. This limiting value or average rate of change of function f (x) is known as instantaneous rate of change.

1.1.2.1 Definition:

The derivative of y = f (x) is defined as

$$\frac{dy}{dx} = \underset{\Delta x \to 0}{Lt} \frac{\Delta y}{\Delta x}$$

i.e. $$\frac{dy}{dx} = \underset{\Delta x \to 0}{Lt} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Note:

1. Derivative $\frac{dy}{dx}$ measures the rate of change of the variable y with respect to the variable x.
2. $\frac{d}{dx}$ doesn't mean 'd' divided by 'dx'. It is a symbol meaning the differential coefficient of the function.
3. The symbol $\frac{dy}{dx}$ is written in many other ways such as $y', y_1, \frac{d}{dx}[f(x)], f'(x), Dy$ etc.
4. The derivative is also called the rate measure.
5. The incremental ratio $\frac{f(x+h) - f(x)}{h}$ is not defined when h = 0.
6. The derivative of f(x) will exist only if $\underset{x \to a}{Lt} \frac{f(x+h) - f(x)}{h}$ exists.

1.1.2.2 Geometric Intepretation of $\frac{dy}{dx}$:

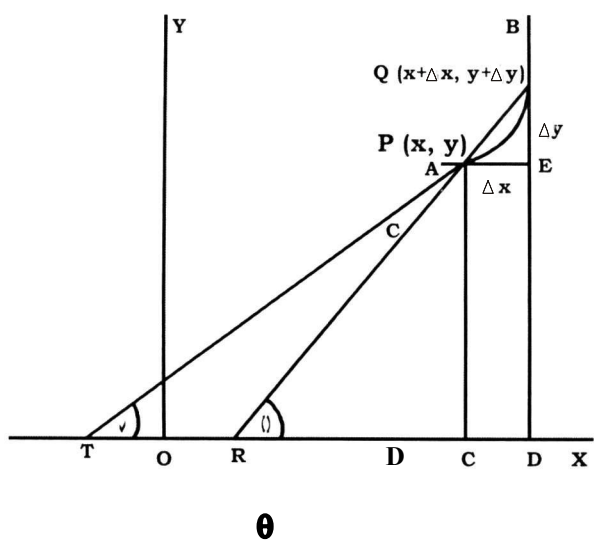
Let y = f (x) be a continuous function of x.
Let P (x, y) be any point on the curve AB.

$Q(x+\Delta x, y+\Delta y)$ be a point in the neighbourhood of P.

Join QP and extend the line to meet x-axis in R.
Draw PC and QD perpendiculars on OX and PE $\perp$ on DQ.

Let, $\angle QRX = \theta$, now $PE = \Delta x, EQ = \Delta y.$

and $\frac{\Delta y}{\Delta x} = \frac{EQ}{PE} = \tan\theta$



Now, as $Q \to P, \Delta x \to 0$ and the limit of $\frac{\Delta y}{\Delta x}$, If it exists, is called the derivative of y w.r.t. x and is expressed as $\underset{\Delta x \to 0}{Lt} \frac{\Delta y}{\Delta x} = \underset{Q \to P}{Lt} \tan\theta =$ Slope of the curve at $P = \frac{dy}{dx}$.

Hence $\frac{dy}{dx}$ at any point P (x, y) on the curve of f (x) is equal to the slope of the tangent at P (x, y).

1.1.2.3 Differentiation 'ab-initio':

The process of obtaining derivatives without making use of the established standard form or theorems on differentiation. The technique of doing it is described as differentiation 'ab-initio'.

Let y = f (x) .....(i)

Hence x is independent variable and y is dependent variable. Suppose, there is a small increment in the value of x and is denoted by Δx, and Δy be increment in y.

$$y + \Delta y = f(x + \Delta x) \qquad .....(ii)$$

Subtracting (i) from (ii), we get

$$\Delta y = f(x + \Delta x) - f(x)$$

Dividing both sides by Δx

$$\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

Take Limits as $\Delta x \to 0$

$$\underset{\Delta x \to 0}{Lt} \frac{\Delta x}{\Delta y} = \underset{\Delta x \to 0}{Lt} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$\frac{dy}{dx}$ or $\underset{\Delta x \to 0}{Lt} \frac{\Delta y}{\Delta x}$ is the derivative or differential coefficient of y with respect to x

$\underset{\Delta x \to 0}{Lt} \frac{\Delta y}{\Delta x}$ or $\frac{dy}{dx}$

Example 1:

Let $y = \frac{1}{x^2}$

Now, we have to find $\frac{dy}{dx}$ from the first principle.

$$y = \frac{1}{x^2} \quad .....(i)$$

Let Δx be an increment in the value of x and Δy be the corresponding increment in the value of y.

$$\therefore \quad y + \Delta y = \frac{1}{(x+\Delta x)^2} \qquad \text{.....(ii)}$$

Subtracting (i) from (ii)

$$\Delta y = \frac{1}{(x+\Delta x)^2} - \frac{1}{x^2} = \frac{x^2 - (x+\Delta x)^2}{x^2\,(x+\Delta x)}$$

$$= \frac{x^2 - x^2 - \Delta x^2 - 2x\Delta x}{(x+\Delta x)^2\, x^2} = -\frac{\Delta x\,(\Delta x + 2x)}{(x+\Delta x)^2\, x^2}$$

Dividing both sides by Δx,

$$\frac{\Delta y}{\Delta x} = -\frac{(\Delta x + 2x)}{(x+\Delta x)^2\, x^2}$$

Take limits as $\Delta x \to 0$

$$\frac{dy}{dx} = -\frac{2x}{x^4} = -\frac{2}{x^3}$$

Example 2:

Let $y = \sqrt{ax+b}$ .....(i)

We have to differentiate (i) by the first principle

$$y + \Delta y = \sqrt{a(x+\Delta x)+b} \qquad \text{.....(ii)}$$

Subtracting (i) from (ii)

$$\Delta y = \sqrt{a(x+\Delta x)+b} - \sqrt{ax+b}$$

Rationalizing

$$\Delta y = \sqrt{ax+a\Delta x+b} - \sqrt{ax+b} \times \frac{\sqrt{ax+a\Delta x+b} + \sqrt{ax+b}}{\sqrt{ax+a\Delta x+b} + \sqrt{ax+b}}$$

$$= \frac{ax+a\Delta x+b-ax-b}{\sqrt{ax+a\Delta x+b} + \sqrt{ax+b}}$$

Dividing both sides by Δx,

$$\frac{\Delta y}{\Delta x} = \frac{a\Delta x}{\Delta x\left[\sqrt{ax + a\Delta x + b} + \sqrt{ax + b}\right]}$$

Take limits as $\Delta x \to 0$

$$\underset{\Delta x \to 0}{Lt} \frac{\Delta y}{\Delta x} = \underset{\Delta x \to 0}{Lt} \frac{a}{\left[\sqrt{ax + a\Delta x + b} + \sqrt{ax + b}\right]} = \frac{a}{\sqrt{ax + b} + \sqrt{ax + b}} = \frac{a}{2\sqrt{ax + b}}$$

$$\frac{dy}{dx} = \frac{a}{2\sqrt{ax + b}}$$

Exercise 1:

Differentiate the following by ab-initio principle.

1. $y = \sqrt{x}$
2. $y = \frac{4x - 3}{2x + 1}$
3. $y = \frac{x}{x + 1}$
4. $y = 2x^2 - 4x + 5$
5. $y = \frac{ax + b}{cx + d}$

1.1.3 Derivatives of Some Standard Functions:

In this section, we will discuss set of rules to find derivatives of some standard functions. We assume that the functions are differentiable.

1.1.3.1 Rules for Simple Functions:

I. Derivatives of Constant:

If y = c, where c is a constant.

Then $\frac{dy}{dx} = \frac{d}{dx}(c) = 0$

Proof:

Now y = c.....(i)

$y + \Delta y = c$ .....(ii)

Subtracting (i) from (ii) $\Delta y = 0$

Dividing by $\Delta x, \frac{\Delta y}{\Delta x} = 0$

Take limits as $\Delta x \to 0$

$$\therefore \quad \underset{\Delta x \to 0}{Lt} \frac{\Delta y}{\Delta x} = 0$$

or $\frac{dy}{dx} = 0$ or $\frac{d}{dx}(c) = 0$

Exercise 3:

If y = f (x) = 7

then $\frac{dy}{dx} = 0$

II. Derivative of x^n w.r.t. x.

Derivative of x^n w.r.t. x will be nx^{n-1}

Proof :

Let $y = x^n$ .....(i)

$y + \Delta y = (x + \Delta x)^n$ .....(ii)

Subtracting (i) from (ii) we get, $\Delta y = (x + \Delta x)^n - x^n$

Dividing both sides by Δx,

$$\therefore \quad \frac{\Delta y}{\Delta x} = \frac{(x + \Delta x)^n - x^n}{\Delta x}$$

$$\frac{\Delta y}{\Delta x} = \frac{1}{\Delta x}.x^n \left[\left(1 + \frac{\Delta x}{x}\right)^n - 1\right]$$

Apply Binomial Theorem (for +ve index)

$$\left[(x + a)^n = x^n + {}^nC_1 x^{n-1} a + {}^nC_2 x^{n-2} a^2 + {}^nC_3 x^{n-3} a^3 + ... + {}^nC_n a^n\right]$$

$$\frac{\Delta y}{\Delta x} = \frac{1}{\Delta x}.x^n \left[\left(1 + n.\frac{\Delta x}{x} + \frac{n(n-1)}{\angle 2}\left(\frac{\Delta x}{x}\right)^2 + ...\right) - 1\right]$$

$$= \frac{1}{\Delta x} \times x^n \left[n \quad \frac{\Delta x}{x} + \frac{n(n-1)}{\angle 2}\left(\frac{\Delta x}{x}\right)^2 + ...\right]$$

$$= \frac{1}{\Delta x} \times x^n \frac{\Delta x}{x}\left[n + \frac{n(n-1)}{\angle 2}\left(\frac{\Delta x}{x}\right) + ...\right]$$

$$= x^{n-1}\left[n + \frac{n(n-1)}{\angle 2}\left(\frac{\Delta x}{x}\right) + ...\right]$$

Take limits as $\Delta x \to 0$

$$\therefore \quad \frac{dy}{dx} = x^{n-1}\left[n + 0 + 0.....\right]$$

$$\frac{dy}{dx} = nx^{n-1}$$

or $$\frac{d}{dx}\left[x^n\right] = nx^{n-1}$$

Example 4:

If $y = x^{17}$

$$\frac{dy}{dx} = 17x^{16}$$

Example 5:

If $y = x^8$

$$\frac{dy}{dx} = 8x^7$$

III. Derivative of $y = (ax + b)^n$ is $n(ax + b)^{n-1} . a$

Let $y = (ax + b)^n$ .....(i)

Change x to $x + \Delta x$ and y to $y + \Delta y$, we have,

$$y + \Delta y = \left[a(x + \Delta x) + b\right]^n$$

$$= \left[ax + a\Delta x + b\right]^n = \left[ax + b + a\Delta x\right]^n$$

$$= (ax + b)^n\left(1 + \frac{a\Delta x}{ax + b}\right)^n \qquad .....(ii)$$

Subtract (i) from (ii), we get

$$\Delta y = (ax+b)^n \left(1+\frac{a\Delta x}{ax+b}\right)^n - (ax+b)^n$$

$$= (ax+b)^n \left[\left(1+\frac{a\Delta x}{ax+b}\right)^n - 1\right]$$

$$= (ax+b)^n \left[\left(1+n.\frac{a\Delta x}{ax+b}+\frac{n(n-1)}{\angle 2}\left(\frac{a\Delta x}{ax+b}\right)^2+.....\right)-1\right]$$

$$= (ax+b)^n \left[n.\frac{a\Delta x}{ax+b}+\frac{n(n-1)}{\angle 2}\left(\frac{a\Delta x}{ax+b}\right)^2+.....\right]$$

$$= (ax+b)^n \frac{a\Delta x}{ax+b}\left[n+\frac{n(n-1)}{\angle 2}\left(\frac{a\Delta x}{ax+b}\right)+.....\right]$$

$$= (ax+b)^{n-1} a\Delta x\left[n+\frac{n(n-1)}{\angle 2}\left(\frac{a\Delta x}{ax+b}\right)+.....\right]$$

Divide both sides by Δx,

$$\therefore \quad \frac{\Delta y}{\Delta x} = \frac{(ax+b)^{n-1} a\Delta x}{\Delta x}\left[n+\frac{n(n-1)}{\angle 2}\left(\frac{a\Delta x}{ax+b}\right)+.....\right]$$

Take limits ax $\Delta x \to 0$

$$\underset{\Delta x \to 0}{Lt} \frac{\Delta y}{\Delta x} = a\,(ax+b)^{n-1}\left[n+0+0.....\right]$$

$$\frac{d}{dx}(ax+b)^n = n(ax+b)^{n-1} \times a = an(ax+b)^{n-1}$$

Example 6:

If y = $(7x + 5)^4$

$$\because \frac{dy}{dx} = \frac{d}{dx}(7x+5)^4 = 4(7x+5)^3 x7$$

$$\frac{dy}{dx} = 28(7x+5)^3$$

Example 7:

$$y = (2x-4)^{-5/2}$$

$\therefore$ $\frac{dy}{dx} = \frac{d}{dx}(2x-4)^{-5/2}$

$$\frac{dy}{dx} = -\frac{5}{2}(2x-4)^{-\frac{7}{2}}(2) = -5(2x-4)^{-\frac{7}{2}}$$

$\mu_{2n+1} = 0 \; (n = 0, 1, 2 \ldots)$

$\mu_1 = \mu_3 = \mu_5 = \ldots = 1$

Cor.:

Differentiation does not affect a multiplicative constant.

$\therefore$ $\frac{d}{dx}[au] = a\frac{d}{dx}(u)$ where a is constant and u = f (x)

Let y = au .....(i)

$\Delta y = a \Delta u$

$\therefore$ $y + \Delta y = a(u + \Delta u) = au + a\Delta u$ .....(ii)

Subtracting (i) from (ii)

Dividing both sides by Δx,

$$\frac{\Delta y}{\Delta x} = a\frac{\Delta u}{\Delta x}$$

Take limits as $\Delta x \to 0$

$$\underset{\Delta x \to 0}{Lt}\frac{\Delta y}{\Delta x} = \frac{dy}{dx} = a\frac{du}{dx}$$

$\therefore$ $\frac{d}{dx}[au] = a\frac{d}{dx}(u)$

Example 8:

If $y = 7x^9$

$$\frac{d}{dx}\left(7x^9\right) = 7\frac{d}{dx}\left(x^9\right) = 7 \times 9x^8 = 63x^8$$

Example 9:

If $y = 6x^7$

$$\frac{d}{dx}\left(6x^7\right) = 6\frac{d}{dx}\left(x^7\right) = 6 \times 7x^6 = 42x^6$$

Exercise 2:

Find derivatives of the following functions:

(a) u $v^r = 5x^3$

(b) $y = 8$

(c) $f(x) = (2x - 7)^{-8/3}$

1.1.3.2 Sum Rule/Difference Rule:

$\frac{d}{dx}(a \pm u) = \frac{d}{dx}(u) \pm \frac{d}{dx}(v)$ where u and v are derivable functions at x.

The derivative of the algebraic sum/difference of two functions is equal to the corresponding algebraic sum of their derivatives, provided these derivatives exist.

Let $y = (u + v)$.....(i)

$\therefore \quad y + \Delta y = \left[(u + \Delta u) + (v + \Delta v)\right]$.....(ii)

Subtracting (i) from (ii) $\Delta y = \Delta u + \Delta v$

Dividing both sides by Δx, we get

$$\frac{\Delta y}{\Delta x} = \frac{\Delta u}{\Delta x} + \frac{\Delta v}{\Delta x}$$

Take limits as $\Delta x \to 0$, we get

$$\frac{dy}{dx} = \frac{d}{dx}(u) + \frac{d}{dx}(v)$$

Hence $\frac{d}{dx}(u + v) = \frac{d}{dx}(u) + \frac{d}{dx}(v)$

Similarly we can prove $\frac{d}{dx}(u - v) = \frac{d}{dx}(u) - \frac{d}{dx}(v)$

Generalisation:

Let $y = u_1 \pm u_2 \pm u_3 \pm u_4 \pm \ldots\ldots\ldots\ldots \pm u_n$

Where $u_1, u_2, \ldots\ldots\ldots\ldots\ldots\ldots\ldots u_n$ are derivable function of x, then

$$\frac{dy}{dx} = \frac{du_1}{dx} \pm \frac{du_2}{dx} \pm \frac{du_3}{dx} \pm \frac{du_4}{dx} \pm \ldots\ldots\ldots\ldots\ldots\ldots \pm \frac{du_n}{dx}$$

Example 10:

Let $y = 6x^4 + \frac{1}{2}x^3 - x^2 + 4x - 8$

we have to find $\frac{dy}{dx}$

$$\frac{dy}{dx} = \frac{d}{dx}\left(6x^4 + \frac{1}{2}x^3 - x^2 + 4x - 8\right)$$

$$\frac{dy}{dx} = \frac{d}{dx}\left(6x^4\right) + \frac{1}{2}\frac{d}{dx}\left(x^3\right) - \frac{d}{dx}\left(x^2\right) + \frac{d}{dx}(4x) - \frac{d}{dx}(8)$$

$$= 6\frac{d}{dx}\left(x^4\right) + \frac{3}{2}\left(x^2\right) - 2x + 4\frac{d}{dx}(x) - 0$$

$$\therefore \quad \frac{dy}{dx} = 24x^3 + \frac{3}{2}x^2 - 2x + 4$$

Example 11:

Let $y = 4x^4 - 3x^2 + 2$

$$\frac{dy}{dx} = \frac{d}{dx}\left(4x^4\right) - \frac{d}{dx}\left(3x^2\right) + \frac{d}{dx}(2)$$

$$= 4\frac{d}{dx}\left(x^4\right) - 3\frac{d}{dx}\left(x^2\right) + 0 = 16x^3 - 6x$$

$$= \frac{d}{dx}\left(4x^4 - 3x^2 + 2\right) = 16x^3 - 6x$$

1.1.3.3 Product Rule

Let u and v are two functions of x then

$$\frac{d}{dx}(uv) = u\frac{d}{dx}(v) + v\frac{d}{dx}(u)$$

Derivative of the product of two functions = first function × derivative of second function + second function × derivative of the first function.

Let y = uv

Let Δx be increment of x and Δu, Δv, Δy be the corresponding increments of u,v and y respectively.

$$y + \Delta y = (u + \Delta u)(v + \Delta v)$$

$$y + \Delta y = uv + v\Delta u + u\Delta v + \Delta u.\Delta v \qquad \text{.....(ii)}$$

Subtracting (i) from (ii)

$$\Delta y = v\Delta u + u\Delta v + \Delta u.\Delta v$$

Dividing both sides by Δx

$$\frac{\Delta y}{\Delta x} = \frac{v\Delta u + u\Delta v + \Delta u.\Delta v}{\Delta x}$$

$$= v\frac{\Delta u}{\Delta x} + u\frac{\Delta v}{\Delta x} + \Delta u\frac{\Delta v}{\Delta x}$$

Take limits as $\Delta x \to 0$

$$\underset{\Delta x \to 0}{Lt}\frac{\Delta y}{\Delta x} = \underset{\Delta x \to 0}{Lt}\left(v\frac{\Delta u}{\Delta x} + u\frac{\Delta v}{\Delta x} + \Delta u\frac{\Delta v}{\Delta x}\right)$$

$$\frac{dy}{dx} = \underset{\Delta x \to 0}{Lt}\, v\frac{\Delta u}{\Delta x} + \underset{\Delta x \to 0}{Lt}\, u\frac{\Delta v}{\Delta x} + \underset{\Delta x \to 0}{Lt}\, \Delta u\frac{\Delta v}{\Delta x} \qquad \left(\frac{dy}{dx}\text{etc.}\right)$$

$$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx} + 0\frac{dv}{dx} \qquad \left(\underset{\Delta x \to 0}{Lt}\frac{\Delta u}{\Delta x} = \frac{du}{dx}\right) \left(\underset{\Delta x \to 0}{Lt}\frac{\Delta v}{\Delta x} = \frac{dv}{dx}\right)$$

$$= v\frac{du}{dx} + u\frac{dv}{dx} \qquad \left(\underset{\Delta x \to 0}{Lt}\frac{\Delta v}{\Delta x} = \frac{dv}{dx}\right)$$

$$\therefore \quad \frac{d}{dx}(u\,.\,v) = v\frac{du}{dx} + u\frac{dv}{dx}$$

Generalisation:

Let y = uvw = uv(w)

$$\therefore \quad \frac{dy}{dx} = uv\frac{dw}{dx} + w\frac{d}{dx}(uv)$$

$$= uv\frac{dw}{dx} + w\left(u\frac{dv}{dx} + v\frac{du}{dx}\right)$$

$$\frac{d(u\,v\,w)}{dx} = uv\frac{dw}{dx} + wu\frac{dv}{dx} + wv\frac{du}{dx}$$

Example 12:

Let $y = (x - 4)^2\ (x^2 + 7)$

$$\frac{dy}{dx} = \frac{d}{dx}(x-4)^2\left(x^2+7\right)$$

$$= (x-4)^2\ \frac{d}{dx}\left(x^2+7\right) + \left(x^2+7\right)\frac{d}{dx}(x-4)^2$$

$$\frac{dy}{dx} = (x-4)^2\ (2x) + \left(x^2+7\right)2\,(x-4)$$

$$= (x-4)^2\ (2x) + 2\,(x-4)\left(x^2+7\right)$$

$$\frac{dy}{dx} = (x-4)(4x^2 - 8x + 14)$$

Example 13:

Let $y = (6x^4 + 9x)\ (3x^2 + 5)$

$$\therefore \quad \frac{dy}{dx} = \frac{d}{dx}\left[\left(6x^4+9x\right)\left(3x^2+5\right)\right]$$

$$= \left(6x^4+9x\right)\frac{d}{dx}\left(3x^2+5\right) + \left(3x^2+5\right)\frac{d}{dx}\left(6x^4+9x\right)$$

$$\frac{dy}{dx} = \left(6x^4+9x\right)(6x) + \left(3x^2+5\right)\left(24x^3+9\right)$$

1.1.3.4 Quotient Rule

If $y = \frac{u}{v}$, where u and v are derivable function of x, then

$$\frac{dy}{dx} = \frac{d^{\left(\frac{u}{v}\right)}}{dx} = \frac{v.\frac{du}{dx} - u.\frac{dv}{dx}}{v^2}$$

Now $y = \frac{u}{v}$ (given) .....(i)

Change x to $x + \Delta x$ and y to $y + \Delta y$, we have

$v\Delta u$

$$y + \Delta y = \frac{u + \Delta u}{v + \Delta v} \qquad \text{.....(ii)}$$

Subtracting (i) from (ii)

$$\Delta y = \frac{u + \Delta u}{v + \Delta v} - \frac{u}{v}$$

$$\Delta y = \frac{v(u + \Delta u) - u(v + \Delta v)}{v(v + \Delta v)}$$

$$\Delta y = \frac{vu + u\Delta v - uv - u\Delta v}{v(v + \Delta v) \quad v} = \frac{v\Delta u - u\Delta v}{v(v + \Delta v)}$$

Dividing both sides by Δx,

$$\frac{\Delta y}{\Delta x} = \frac{\frac{v\Delta u - u\Delta v}{v(v + \Delta v)}}{\Delta x} = \frac{v\frac{\Delta u}{\Delta x} - u\frac{\Delta v}{\Delta x}}{v^2 + v\Delta v}$$

Take limits as $\Delta x \to 0$

$$\underset{\Delta x \to 0}{\text{Lt}} \frac{\Delta y}{\Delta x} = \underset{\Delta x \to 0}{\text{Lt}} \frac{v\frac{\Delta u}{\Delta x} - u\frac{\Delta v}{\Delta x}}{v^2 + v\Delta u}$$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$\frac{d}{dx}$ (Quotient of two functions)

$$= \frac{Deno\min ator \times \frac{d}{dx}(Numerator) - Numerator \times \frac{d}{dx}(Deno\min ator)}{(Deno\min ator)^2}$$

Example 14:

Let $y = \dfrac{x^3 - 6x^2}{5x^2 - 1}$

$$\therefore \quad \frac{dy}{dx} = \frac{\left(5x^2 - 1\right)\frac{d}{dx}\left(x^3 - 6x^2\right) - \left(x^3 - 6x^2\right)\frac{d}{dx}\left(5x^2 - 1\right)}{\left(5x^2 - 1\right)^2}$$

$$\therefore \quad \frac{dy}{dx} = \frac{\left(5x^2 - 1\right)\left(3x^2 - 12x\right) - \left(x^3 - 6x^2\right)(10x)}{\left(5x^2 - 1\right)^2}$$

$$\frac{dy}{dx} = \frac{5x^4 - 3x^2 + 12x}{\left(5x^2 - 1\right)^2}$$

$$-(1-x)^{\frac{1}{2}}.\frac{d}{dx}(1+x)^{-\frac{1}{2}}$$

Example 15:

Let $y = \sqrt{\dfrac{1-x}{1+x}}$

$$\therefore \quad \frac{dy}{dx} = \frac{(1+x)^{\frac{1}{2}}\frac{d}{dx}(1-x)^{\frac{1}{2}} - (1-x)^{\frac{1}{2}}\frac{d}{dx}(1+x)^{\frac{1}{2}}}{(\sqrt{1+x})^2}$$

$$= \frac{\sqrt{1+x}\left[\frac{1}{2}(1-x)^{-\frac{1}{2}}.(-1)\right] + (1-x)^{\frac{1}{2}}.\frac{1}{2}(1+x)^{-\frac{1}{2}}}{(1+x)}$$

$$\frac{dy}{dx} = \frac{-\frac{1}{2}\frac{\sqrt{1+x}}{\sqrt{1-x}} - \frac{1}{2}\frac{\sqrt{(1-x)}}{\sqrt{1+x}}}{1+x}$$

$$\frac{dy}{dx} = \frac{-\frac{1}{2}\frac{(1+x)+(1-x)}{\sqrt{1-x}\sqrt{1+x}}}{1+x} = \frac{-1}{(1-x)\sqrt{1-x^2}}$$

1.1.3.5 Chain Rule/Function of a Function Rule

If y = f (u) is derivable at u and u = ϕ (x) is derivable at x, then y is derivable at x and

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

Let Δx, be an increment of x and Δu be the corresponding increment in u and the corresponding increment in y will be Δy

$$\frac{\Delta y}{\Delta x} = \frac{\Delta y}{\Delta u} \cdot \frac{\Delta u}{\Delta x}$$

Take limits as $\Delta x \to 0$ and $\Delta u \to 0$ we get,

$$\underset{\Delta x \to 0}{Lt} \frac{\Delta y}{\Delta x} = \underset{\Delta x \to 0}{Lt} \frac{\Delta y}{\Delta u} \times \underset{\Delta x \to 0}{Lt} \frac{\Delta u}{\Delta x}$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

Generalisation:

If $y = \phi(u), f = f(z), z = \Psi(x)$

Then $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dz} \cdot \frac{dz}{dx}$

Example 16:

Let y = $2u^2 + 3$ and u = $5x^3$

To find $\frac{dy}{dx}$, we have to find $\frac{dy}{du}$ and $\frac{du}{dx}$

Now y = $2u^2 + 3$

$\therefore \quad \frac{dy}{dx} = 4u$

u = $5x^3$

$\therefore \quad \frac{du}{dx} = 15x^2$

$\therefore \quad \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = 4u \times 15x^2 = 60ux^2$

$\therefore \quad \frac{dy}{dx} = 60(5x^3)x^2 = 300x^5$

Example 17:

Let $y = t^2 at^3$ and $t = bx^2$

We have to find $\frac{dy}{dx}$

Now $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$

$y = at^3$, and $t = bx^2$

$\therefore \quad \frac{dy}{dt} = 3a^2 \qquad \therefore \quad \frac{dt}{dx} = 2bx$

$\therefore \quad \frac{dy}{dx} = 3at^2 \times 2bx = 6abxt^2 = 6abx(bx^2)^2$

$\therefore \quad \frac{dy}{dx} = 6ab^3x^5$

Exercise 3:

1. Differentiate the following functions w.r.t. x
   (i) $(x^3 + 2x^2)^3$
   (ii) $\sqrt{\frac{x^2 - 2ax}{a^2 - 2ab}}$
   (iii) $(2x^4 + 1)^{1/2} + (x + 1)^{1/2}$
   (iv) $(x^2 + 1)(3x^2 - 2x^2)$
2. Find the derivative of y with respect to x when
   (i) $y = 2u^2 - 3$ and
   $u = \frac{1}{x^2}$,
   (ii) $y = 3t^2 + 1$ and
   $t = u^2 + u$, $u = x$

1.1.4. Differentiation of Implicit Functions

When y is defined implicitly as a function of x then it is called implicit function.

For example, $4x^2y - 7x + 8y = 0$ is an implicit function. Here derivative of y w.r.t. x may be found by considering y as a function of x and differentiating

term by term.

We make use of the following results.

$$\frac{d}{dx}(x^n) = nx^{n-1}, \frac{d}{dx},(y^n) = ny^{n-1}\frac{d}{dx}(y) \qquad +y^2$$

Example 18: 2

Consider $x^3 - 2x^2y + 3xy^2 - 25 = 0$

$\therefore$

$$\frac{d}{dx}x^3 - 2(y\frac{d}{dx}x^2 + x^2\frac{dy}{dx}) + 3\left[x\frac{d}{dx}y2\frac{d}{dx}x\right] - \frac{d}{dx}(25) = 0$$

$$3x^2 - 2\left[2xy + x\frac{dy^2}{dx}\right] + 3\left[2xy\frac{dy}{dx} + y^2\right] - 0 = 0$$

$$3x^2 - 4xy - 2x^2\frac{dy}{dx} + 6xy\frac{dy}{dx} + 3y^2 = 0$$

$$\left[-2x^2 + 6xy\right]\frac{dy}{dx} = -3y^2 - 3x^2 - 3x^2 + 4xy$$

$$\left(6. \quad 2x^2\right)\frac{dy}{dx} = -3(y^2 + x^2) + 4xy$$

xy $$\frac{4xy - 3(x^2 + y^2)}{6xy - 2x^2}$$

Example 19:

Let $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

$\therefore \quad \frac{d}{dx}\left[ax^2 + 2hy + by^2 + 2gx + 2fy + c\right] = 0$

$\therefore \quad 2ax^2 + 2h\frac{d}{dx}(xy) + b\frac{d}{dx}(y^2) + 2\frac{d}{dx}(gx) + 2\frac{d}{dx}(fy) = 0$

$$2ax + 2h\left[y + x.\frac{dy}{dx}\right] + b\left(2y\frac{dy}{dx}\right) + 2g + 2f.\frac{dy}{dx} = 0$$

$$2ax + 2hy + 2g + 2hx\frac{dy}{dx} + 2by\frac{dy}{dx} + 2f.\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-2[ax + hy + g]}{2[hx + by + f]} = \frac{-(ax + hy + g)}{hx + by + f}$$

Example 20:

Find if $y = \sqrt{x + \sqrt{x + \sqrt{x + ...\infty}}}$

$$y = \sqrt{x + \sqrt{x + \sqrt{x + ...\infty}}}$$

$y = \sqrt{x + y}$ squaring both sides, we get

$y^2 = x + y$

$\therefore$ $x = y^2 - y$

Diff. both sides w.r.t. y, (considering x as a function of y)

$$\frac{dx}{dy} = 2y - 1$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{2y - 1}$$

1.1.5 Differentiation of Parametric Equations

If x = f(t) and y = $\phi(t)$, x and y both are functions of variable t, then these equations are known as parametric equations.

Now $\frac{dy}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt}$

Example 21:

Let $x = t^4 + 3t^2 + t$, $y = 7t^2 + 6t - 9$

For finding $\frac{dy}{dx}$ we have to find $\frac{dy}{dt} \div \frac{dx}{dt}$

Now $\frac{dy}{dt} = 14t + 6, \frac{dx}{dt} = 4t^3 + 6t + 1$

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt} = \frac{14t + 6}{4t^3 + 6t + 1}$$

Exercise 4:

Find $\frac{dy}{dx}$, where

(i) $x^2y + xy^2 = 25$

(ii) $x^2 + 3xy + y^2 = 4$

(iii) $xy^3 + x^3y = a^3$

(iv) $x = 4t^2 + 6t + 1, \quad y = 7t - 1$

(v) $x = \frac{2t+3}{t^2-1}, \quad y = \frac{3t-2}{t^2-1}$

1.1.6 Summary :

In this lesson you have studied the meaning and definition of derivative. You have also learnt about the geometrical interpetation of Derivative. Two methods of finding derivatives using First Principle and using Standard Rules have also been explained with illustrations.

1.1.7 Key Words:

Limit : The method of knowing the behaviour of a function Y = f(x) as the independent variable x approaches some particular value.

Derivative : A function that expresses the slope of another function at every point.

Slope : The rate of change in the dependent variable (y) for a unit change in the independent variable (x).

Differentiation : The process of finding derivatives.

1.1.8 Suggested Readings:

1. S.C. Aggarwal and R.K. Rana : Basic Mathematics for Economists
2. B.M. Aggarwal : Mathematics for Business and Economics
3. Aggarwal and Joshi : Mathematics for Students of Economics

1.1.9 List of Questions:

1.1.9.1 Short Questions

1. Find out derivatives of the following functions w.r.t. x.

(i) $x^3 + 7x + 2$ (ii) $y = \frac{6}{x^{3/2}}$ (iii) $y = x^{-9/2}$ (iv) $Y = 10x^4$ (v) $y = \sqrt{x} + \frac{1}{\sqrt{x}}$

1.1.9.2 Long Questions

(i) Find out derivative of Y w.r.t. x

Given $Y = \frac{4+u}{3-u}$ and $U = \frac{2x}{1-x^2}$

(ii) If $Y = 2w^2 + 1$, $w = 2z^2$ and $z = 2x + 3x^2$
Find out derivative of Y w.r.t. x

(iii) If $Y = x^2 + \frac{1}{x^2}$, show that

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} = 4y$$

(iv) Differentiate from the first Principle

(a) $\sqrt{\frac{x+1}{x+2}}$ (b) $\frac{1}{\sqrt{ax+b}}$

(v) Explain geometric meaning of the derivative.

M.A. (ECONOMICS) PART–I **PAPER–III**

(BASIC QUANTITATIVE METHODS)

LESSON NO. 1.2 **AUTHOR : DR. VIPLA CHOPRA**

DIFFERENTIATION
(LOGARITHMIC AND EXPONENTIAL FUNCTIONS)

Structure

1.2.0 Introduction:

In Lesson No. 1, we have studied the derivatives of simple functions. In the present lesson, we will study the derivatives of Logarithmic and Exponential Functions. Some elementary knowledge of successive derivatives is also given. This lesson has been divided into three parts. In the first part, derivatives of standard logarithmic functions will be studied. Second part is concerned with derivatives of some standard exponential functions. Successive derivatives or higher order derivatives form the third part of this lesson.

1.2.1 Objectives:

After going through this lesson you will be able to

* learn about the derivatives of logarithmic functions.
* obtain the derivatives of exponential functions.
* distinguish between simple derivatives and successive derivatives.

1.2.2 Derivatives of Logarithmic Functions:

For understanding derivatives of Logarithmic functions the students must have to be familiar with logs and anti-logs. Three rules of Logarithm are also important to remember which are as follows :

(i) Log(mn) = Log m + Log n

(ii) $\text{Log}\left(\frac{m}{n}\right) = \text{Log } m - \text{Log } n$

(iii) Log m^n = n Log m

1.2.2.1 Definition:

$y = \log_a x$ is called a logarithmic function. Here, the base of logarithm is 'a'.

$y = \log_e x$ is known as natural logarithmic function as 'e' is the base. It can also be written as log x.

It represents the standard form of logarithmic function.

The derivatives of such standard logarithmic functions lead to standard results.

1.2.1.2 Derivative of $y = \log_a x$

Given $y = \log_a x$ .....(i)

$y + \Delta y = \log_a (x + \Delta x$ .....(ii)

Subtracting (i) from (ii), we get

$y + \Delta y = \log_a (x + \Delta x)$ -$\log_a x$

$$= \log_a \frac{(x + \Delta x)}{x} \qquad \left[\log m - \log n = \log \frac{m}{n}\right]$$

$$= \log_a \left(1 + \frac{\Delta x}{x}\right)$$

Divide both sides by Δx, we get

$$\frac{\Delta y}{\Delta x} = \frac{1}{\Delta x} \log_a \left(1 + \frac{\Delta x}{x}\right)$$

$$= \log_a \left(1 + \frac{\Delta x}{x}\right)^{1/\Delta x} = \log_a \left[\left(1 + \frac{\Delta x}{x}\right)^{x/\Delta x}\right]^{1/x}$$

$$\frac{\Delta y}{\Delta x} = \frac{1}{x}\log_a\left(1+\frac{\Delta x}{x}\right)^{x/\Delta x} \quad (\therefore \log_a m^n = n\log_a m)$$

Take limits as $\Delta x \to 0$

$$\frac{dy}{dx} = \frac{1}{x}\underset{\Delta x\to 0}{\text{Lt}}\log_a\left(1+\frac{\Delta x}{x}\right)^{\frac{x}{\Delta x}}$$

$$= \frac{1}{x}\log_a e \qquad \left[\underset{\Delta x\to 0}{\text{Lt}}\left(1+\frac{\Delta x}{x}\right)^{\frac{x}{\Delta x}} = e\right]$$

Hence $\frac{d}{dx}(\log_a x) = \frac{1}{x}\log_a e$ .....(iii)

Example 1:

Let $y = \log_5 x$

$\therefore \quad \frac{dy}{dx} = \frac{1}{x}\log_5 e$

Example 2:

If $y = \log_a x^2$

then $y = 2\log_a x \qquad (\therefore \log_a m^n = n\log_a m)$

$\therefore \quad \frac{dy}{dx} = 2\frac{d}{dx}(\log_a x)$

$\therefore \quad \frac{dy}{dx} = \frac{2}{x}\log_a e$

Cor. I:

When $y = \log_e x$

In order to find its derivative put a = e in (iii)

$$\frac{d}{dx}\log_e x = \frac{1}{x}\log_e e = \frac{1}{x} \qquad (\log_e e = 1)$$

or $\frac{d}{dx}(\log x) = \frac{1}{x}$ [When base is not mentioned, it is understood to be 'e']

Example 3:

Let $y = \log_e x^3$

$y = 3 \log_e x \quad (\therefore \log m^n = n \log m)$

$\therefore \quad \frac{dy}{dx} = \frac{3}{x}$

Example 4:

Let $y = \log \sqrt{x}$

$\therefore \quad y = \frac{1}{2} \log x$

$\therefore \quad \frac{dy}{dx} = \frac{1}{2} \times \frac{1}{x} = \frac{1}{2x}$

1.2.2.3 Function of Function Rule

Let $y = \log_a u$ where u is derivable function at x.

By Chain Rule, we know,

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

Following this Rule, we get

$$\frac{d}{dx}(y) = \frac{d}{dx}(\log u)\frac{du}{dx} = \frac{1}{u}\log_a e \frac{du}{dx}$$

Example 5:

Let $y = \log_a (x^3 + 1)$

$\therefore \quad \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

Put $u = x^3 + 1 \qquad \therefore \frac{du}{dx} = 3x^2$

Now $y = \log_a u$

$\therefore \quad \frac{d}{dx}(\log_a u) = \frac{1}{u}\log_a e \times \frac{du}{dx} = 3x^2 \times \frac{1}{x^3+1}\log_a e$

$\therefore \quad \frac{d}{dx}\log_a (x^3+1) = \frac{3x^2}{x^3+1}\log_a e$

or When y = log u, where u is the function of x.

then $\frac{dy}{dx} = \frac{1}{u}\log_e e \times \frac{du}{dx} = \frac{1}{u} \times \frac{du}{dx} \qquad (\log_e e = 1)$

Example 6:

Let y = log $\sqrt{1-x^3}$

We have to differentiate w.r.t. x

$$\therefore \quad \frac{dy}{dx} = \frac{d}{dx}\log(1-x^3)^{1/2} = \frac{1}{2}\frac{d}{dx}\log(1-x^3)$$

$$\therefore \quad \frac{d}{dx}\log(\sqrt{1-x^3}) = \frac{1}{2(1-x^3)} \times (-3x^2)$$

$$= -\frac{3}{2}\frac{x^2}{(1-x^3)}$$

Example 7:

Let y = log $\sqrt{\frac{x+1}{x-1}}$

$$y = \log\left(\frac{x+1}{x-1}\right)^{\frac{1}{2}} = \frac{1}{2}\log\left(\frac{x+1}{x-1}\right) \qquad \left(\log\frac{m}{n} = \log m - \log n\right)$$

$$\therefore \quad y = \frac{1}{2}[\log(x+1) - \log(x-1)]$$

$$\frac{dy}{dx} = \frac{1}{2}\left[\frac{1}{x+1} - \frac{1}{x-1}\right] = \frac{1}{2}\left[\frac{x-1-x-1}{x^2-1}\right]$$

$$= \frac{1}{2} \times \frac{-2}{x^2-1}$$

$$= -\frac{1}{x^2-1}$$

1.2.2.4 Product Rule:

Some functions involving the product of two or more variables may be complicated. Logarithmic derivation becomes very helpful in solving such problems.

Let y = uv

where u and v are derivable functions at x.

Take log on both sides, we get

log y = log u + log v

$$\therefore \quad \frac{d}{dx}\log y = \frac{d}{dx}\log u + \frac{d}{dx}\log v$$

$$\therefore \quad \frac{1}{y}\times\frac{dy}{dx} = \frac{1}{u}\times\frac{du}{dx} + \frac{1}{v}\times\frac{dv}{dx}$$

$$\therefore \quad \frac{dy}{dx}(uv) = \frac{d}{dx}(uv) = uv\left[\frac{1}{u}.\frac{du}{dx} + \frac{1}{v}.\frac{dv}{dx}\right] = v\frac{du}{dx} + u\frac{dv}{dx}$$

Example 8:

Let $y = (2x^2 + 1)(x + 1)$

$\log y = \log[(2x^2 + 1)(x + 1)]$

$= \log(2x^2 + 1) + \log(x + 1)$ [log mn = log m + log n]

Take derivative on both sides, we get

$$\frac{1}{y}.\frac{dy}{dx} = \frac{1}{(2x^2+1)}.\frac{d}{dx}(2x^2+1) + \frac{1}{(x+1)}\frac{d}{dx}(x+1)$$

$$\frac{1}{y}\times\frac{dy}{dx} = \frac{4x}{(2x^2+1)} + \frac{1}{(x+1)}$$

$$\therefore \quad \frac{d}{dx}[(2x^2+1)(x+1)] = (2x^2+1)(x+1) \qquad \left[\frac{4x^2+4x+2x^2+1}{(2x^2+1)(x+1)}\right] = 6x^2+4x+1$$

Example 9:

Let $y = (4x - 1)^2\sqrt{x+4}$

Take log on both sides, we get

$\log y = \log\left[(4x-1)^2\sqrt{x+4}\right]$

$\log y = \log(4x - 1)^2 + \log\sqrt{x+4}$ [log mn = log m + log n]

$\log y = 2\log(4x-1) + \frac{1}{2}\log(x+4)$ [log mn = n log m]

Differentiate w.r.t. x

$$\therefore \quad \frac{1}{y}\times\frac{dy}{dx} = 2\frac{4}{4x-1} + \frac{1}{2}\times\frac{1}{x+4}\times 1$$

$$\frac{1}{y}.\frac{dy}{dx} = \frac{8}{4x-1} + \frac{1}{2(x+4)}$$

$$\therefore \quad \frac{dy}{dx} = \left(4x^2 - 1\right)\sqrt[2]{x+4}\left[\frac{8}{4x-1} + \frac{1}{2(x+4)}\right]$$

$$= (4x-1)^2\ \sqrt[2]{x+4}\left[\frac{(20x+63)}{2(4x-1)(x+4)}\right]$$

$$\frac{dy}{dx} = \frac{(4x-1)}{2\sqrt{x+4}}\left[20x+63\right] = \frac{(4x-1)(20x+63)}{2\sqrt{x+4}}$$

1.2.2.5 Quotient Rule

Let $y = \frac{u}{v}$ where u and v are derivable functions at x.

Take log on both sides,

$$\log y = \log\left(\frac{u}{v}\right) = \log u - \log v$$

$$\frac{d}{dx}\log y = \frac{d}{dx}\log u - \frac{d}{dx}\log v$$

$$\frac{1}{y} \times \frac{dy}{dx} = \frac{1}{u}\frac{du}{dx} - \frac{1}{v}\frac{dv}{dx}$$

or $$\frac{dy}{dx} = y\left[\frac{1}{u} \times \frac{du}{dx} - \frac{1}{v} \times \frac{dv}{dx}\right]$$

$$\therefore \quad \frac{d}{dx}\left(\frac{u}{v}\right) = \frac{u}{v}\left[\frac{1}{u} \times \frac{du}{dx} - \frac{1}{v} \times \frac{dv}{dx}\right]$$

Example 10:

Let $y = \frac{x}{(x+3)(x+4)}$

Let u = x, v = (x + 3) (x + 4), y = $\frac{u}{v}$

Take log on both sides,

$$\therefore \quad \log y = \log\left[\frac{x}{(x-3)(x+4)}\right] \qquad \log y = \log x - \log (x + 3)\ (x + 4)$$

log y = log x – [log (x + 3) + log (x + 4)]

Differentiate w.r.t. x

$$\frac{1}{y}\cdot\frac{dy}{dx}=\frac{1}{x}-\left[\frac{1}{x+3}+\frac{1}{x+4}\right]$$

$$\frac{1}{y}\times\frac{dy}{dx}=\frac{1}{x}-\frac{2x+7}{(x+3)(x+4)}=\frac{x^2+7x+12-2x^2-7x}{x(x+3)(x+4)}$$

$$\frac{1}{y}\cdot\frac{dy}{dx}=\frac{-x^2+12}{x(x+3)(x+4)}$$

$$\therefore \quad \frac{dy}{dx}=y\left[\frac{-x^2+12}{x(x+3)(x+4)}\right]$$

$$\therefore \quad \frac{dy}{dx}=\frac{x}{(x+3)(x+4)}\times\frac{-x^2+12}{x(x+3)(x+4)}=\frac{-(x^2-12)}{(x+3)^2(x+4)^2}$$

$$\therefore \quad \frac{dy}{dx}=\frac{-(x^2-12)}{(x+3)^2(x+4)^2}$$

Example 11:

Let $y=\frac{x^3}{(x+5)^2}$

$\therefore$ $\log y = \log x^3 - \log (x + 5)^2$

$\log y = 3 \log x - 2 \log (x + 5)$

Differentiate w.r.t. to x, we get

$$\frac{1}{y}\times\frac{dy}{dx}=\frac{3}{x}-\frac{2}{x+5} \quad \therefore \frac{dy}{dx}=\frac{x^3}{(x+5)^2}\left[\frac{x+15}{x(x+5)}\right]=\frac{x^2(x+15)}{(x+5)^3}$$

Exercise 1:

1. Find the derivative of the following functions w.r.t. to x

   (i) $\log_a (x^3 + 1)$ (ii) $\log (4 - 2x^3)$ (iii) $\log\frac{1+x^2}{1-x}$

2. Differentiate the following :

   (i) $y=\log\sqrt{a^2+x^2-x}$

   (ii) $y = \log (x + 2) (x^3 - x)$

3. If $x = y (1 + \log x)$, show that

   $$\frac{dy}{dx}=\frac{\log x}{(1+\log x)^2}$$

4. If $y = \frac{(x+2)+(x+3)}{(x-2)(x+3)}$, find $\frac{dy}{dx}$

5. If $y = \sqrt{\frac{1-x}{1+x}}$, show that

$$(1-x)^2 \frac{dy}{dx} + y = 0$$

6. If $\log_a (3x^2 + 4x + 5)$, find derivative of y with respect to x.
7. Find the differential coefficient of the following function:

$$\log\left[x + \sqrt{x^2 + a^2}\right]$$

8. Find the differential coefficient of x^x.

1.2.3 Derivatives of Exponential Functions :

In this section, we will study the derivatives of exponential functions. Logarithm will be used to solve these functions.

1.2.3.1 Definition

Consider $x = a^y$. It is an exponential function. Similarly, $y = e^x$ is known as natural exponential function. It is a standard form of natural exponential function. This function i.e. $y = e^x$ is a single valued continuous and increasing function.

1.2.3.2 Differential Coefficient of a^x

Let $y = a^x$ .....(i)

Take Log of both sides, we get

$$\therefore \quad \frac{d}{dx}(\log y) = \frac{d}{dx}(x.\log a) = \log a.\frac{d}{dx}(x)$$

$$\therefore \quad \frac{1}{y}\frac{dy}{dx} = \log a.1$$

$$\frac{dy}{dx} = y\log a = a^x \log a$$

$$\therefore \quad \frac{dy}{dx}(a^x) = a^x.\log a$$

Cor.:

Put a = e in (i), we get

$y = e^x$

$$\therefore \quad \frac{dy}{dx} = \frac{d}{dx}(e^x) = e^x.\log e = e^x \qquad (\log e = 1)$$

Thus $\frac{d}{dx}(e^x) = e^x$

Cor.:

If $y = a^u$ where u is function of x.

By Chain Rule, we know that

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \quad \text{.....(ii)}$$

Now $y = a^u$

$$\therefore \quad \frac{dy}{du} = \frac{d}{du}(a^u) = a^u \log a$$

Put in (ii)

$$\therefore \quad \frac{dy}{dx} = \frac{d}{du}(a^u) = a^u \log a . \frac{du}{dx}$$

Cor.:

If $y = e^u$ where u is the function of x.

$$\frac{dy}{dx} = \frac{d}{dx}(e^u) = e^u \log \frac{du}{dx} = e^u \frac{du}{dx}$$

$$\therefore \quad \frac{d}{dx}(e^u) = e^u \frac{du}{dx}$$

1.2.3.3 Function of a Function Rule

Let $y = e^u$ where u is the derivable function at x.

Take logarithms of both sides

$\therefore$ $\log y = \log e^u = u \log e = u$

$$\therefore \quad \frac{1}{y}\frac{dy}{dx} = \frac{du}{dx} \qquad \therefore \frac{d}{dx}(e^u) = e^u \frac{du}{dx}$$

Example 12:

Let $y = 3^{x+2}$

$$\therefore \quad \frac{dy}{dx} = 3^{x+2} \log 3 . 1 = 3^{x+2} \log_e 3$$

Example 13:

Let $y = e^{x^2} + 2x$

Put $x^2 + 2x = u$

$\therefore \quad \frac{dy}{dx} = e^u . \frac{du}{dx}$, Now $\frac{du}{dx} = 2x + 2$

$$\frac{dy}{dx} = e^{x^2+2x} \frac{d}{dx}\left(x^2 + 2x\right) = (2x+2)e^{x^2+2x}$$

Example 14:

Consider $y = a^{\sqrt{x^2+1}}$

Put $\sqrt{x^2+1} = u$

$\therefore \quad \frac{dy}{dx} = \frac{dy}{du} . \frac{du}{dx}$

$\therefore \quad \frac{dy}{dx} = a^{\sqrt{x^2+1}} . \log a . \frac{du}{dx} \qquad (\because \frac{d}{dx}(a^u) = a^u \log a . \frac{du}{dx})^d$

$\therefore \quad \frac{dy}{dx} = a^{\sqrt{x^2+1}} . \log a . \frac{1}{2}\left(x^2+1\right)^{-\frac{1}{2}} (2x)$

$\therefore \quad \frac{dy}{dx} = x . \log a . \frac{a^{\sqrt{x^2+1}}}{\sqrt{x^2+1}}$

$\therefore \quad \frac{dy}{dx} = \log a \; . \; \frac{x \; xa^{\sqrt{x^2+1}}}{\sqrt{x^2+1}}$

Example 15:

Consider $y = e^{x^x}$

Take logarithms of both sides, we get

log y = log e^{x^x} = x^x log e = x^x

Take logarithms again, we have

log (log y) = x log x

$\frac{1}{\log y} \times \frac{1}{y} \times \frac{dy}{dx} = 1 + \log x = \log e + \log x = \log ex$

and $\frac{dy}{dx} = y(\log y)(\log ex) = e^{x^x} . x^x . \log ex$

Exercise 2:

1. Find $\frac{dy}{dx}$ when $y = a^x + x^x$

2. Find $\frac{dy}{dx}$ when $y = (\log x)^{\log x}$

3. Differentiate the following to obtain $\frac{dy}{dx}$

   (i) $y = \log\left[x + \sqrt{x^2 + a^2}\right]$

   (ii) $y = 9^{\sqrt{8x^2+4x+2}}$

   (iii) $y = e^{-x^2/2}$

   (iv) $y = e^{x^2} \log 4x$

4. If $x = y(1 + \log x)$, show that

$$\frac{dy}{dx} = \frac{\log x}{(1+\log x)^2}$$

1.2.4 Successive Derivatives

In this section, we are going to explain successive derivatives.

If $y = f(x)$ be the given function then $\frac{dy}{dx} = f(x)$ or 'y_1' is called first derivatives of y with respect to x. Generally f' (x) is again a function of x and if it is derivable, then the derivative of the first derivative can also be obtained. The derivative of the first derivative is called second derivative and is denoted by $\frac{d^2y}{dx^2}$ or f''(x) or y_2 etc. The derivative of the second derivative if it exists, is known as third derivative. Similarly, we have fourth, fifth and other higher order derivatives. Thus the successive derivatives of f (x) are denoted by

$y_1, y_2, y_3, \ldots\ldots\ldots\ldots\ldots\ldots y_n$

or $\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, \ldots\ldots\ldots\ldots \frac{d^ny}{dx^n}$

or f'(x), f''(x), f'''(x) ……

1.2.4.1 Sum and Difference Rule

Let $y = u \pm v$

where u and v are two single-valued derivable functions of x

$\therefore \quad \frac{dy}{dx} = \frac{du}{dx} \pm \frac{dv}{dx}$

and $\frac{d^2y}{dx^2} = \frac{d^2}{dx^2}(u \pm v) = \frac{d}{dx}\left[\frac{d}{dx}(u \pm v)\right]$

$= \frac{d}{dx}\left[\frac{du}{dx} \pm \frac{dv}{dx}\right]$

$= \frac{d}{dx}\left(\frac{du}{dx}\right) \pm \frac{d}{dx}\left(\frac{dv}{dx}\right) = \frac{d^2u}{dx^2} \pm \frac{d^2v}{dx^2}$

Similarly for any value of x which is positive integer, we have

$\frac{d^n}{dx^n}(u \pm v) = \frac{d^nu}{dx^n} \pm \frac{d^nv}{dx^n}$

1.2.4.2 Product Rule (Leibnitz's Theorem)

Let y = uv, where u and v are functions possessing derivatives of the nth order.

Differentiating both sides

$y_1 = uv_1 + u_1v \qquad \left(\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}\right)$

Again differentiating, we get

$y_2 = uv_2 + v_1u_1 + u_1v_1 + u_2v$

$y_2 = u_2v + u_1v_1 + u_1v_1 + uv_2$

$= 2c_0u_2v + 2c_1u_1v_1 + 2c_2uv_2 \left(nc_0 = \frac{\angle n}{\angle 0 \angle n-1} 0 = 1 \; Here \; 2c_0 = \frac{\angle 2}{\angle 2 - 0 \angle 0}(\angle 0 = 1)\right)$

$y_3 = 3c_0u_3v + 3c_1u_2v_1 + 3c_2u_1v_2 + 3c_3uv_3$

Similarly, other higher order derivatives can also be obtained. m^{th} order derivative

i.e. $y_m = u_mv + mc_1u_{m-1}v_1 + mc_2u_{m-2}v_2 + \ldots\ldots\ldots\ldots + mc_ru_{m-r}v_r + \ldots\ldots\ldots + m_{cm}uv_m$

Thus we can show that this theorem is true for any value of n

$\therefore \quad y_n = u_nv_0 + nc_1u_{n-1}v_1 + nc_2u_{n-2}v_2 + \ldots\ldots\ldots\ldots + nc_ru_{n-r}v_r + \ldots\ldots\ldots\ldots + n_{cn}uv_n$

This theorem is known as <u>Leibnitz's Theorem</u>.

Example 16:

Differentiate w.r.t. to x

Let $Y = 4x^5 - 2x^4 + x^3 + 7x^2 + 4x - 5$

$$\therefore \frac{dy}{dx} = 4\frac{d}{dx}(x^5) - 2\frac{d}{dx}(x^4) + \frac{d}{dx}(x^3) + 7\frac{d}{dx}(x^2) + 4\frac{d}{dx}(x) - \frac{d}{dx}(5)$$

$$\frac{dy}{dx} = 20x^4 - 8x^3 + 3x^2 + 14x + 4$$

Differentiate it again

$$\therefore \frac{d^2y}{dx^2} = 80x^3 - 24x^2 + 6x + 14$$

Example 17:

Let y = x log x

$$\therefore \frac{dy}{dx} = x.\frac{d}{dx}(\log x) + \log x\frac{d}{dx}(x)$$

$$\therefore \frac{dy}{dx} = x.\frac{1}{x} + \log x$$

$$\frac{dy}{dx} = 1 + \log x = \log e + \log x = \log(ex)$$

Again differentiating w.r. to x.

$$\frac{d^2y}{dx^2} = \frac{1}{ex}.\frac{d}{dx}(ex) = \frac{e}{ex} = \frac{1}{x}$$

Exercise 3:

1. Find y_2 where

   (i) $y = \frac{2x+5}{x^2-3x+5}$ (ii) $y = x^3e^x$

2. If xy = a + bx, show that

   $$x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 0$$

3. If $y = \frac{\log x}{x}$, find $\frac{d^2y}{dx^2}$

4. If $x^2 + \frac{1}{x^2}$, show that

   $$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - 4y = 0$$

1.2.5 Summary

In this lesson you have studied the meaning of derivative of logarithmic function and different rules for finding derivatives of such functions along with illustrations. Different rules for finding derivatives of different exponential functions have also been explained. In the end some elementary knowledge about successive derivatives has also been given.

1.2.6 Key Words

Functions : It is the rule of correspondence between dependent variable and independent variable (s) so that for every assigned value to the independent variable, the corresponding unique value for the dependent variable is determined.

Logarithmic Function : The inverse of exponential function is called a logarithmic function.

Exponential Function : If the independent variable in any functional relationship appears as exponent (or power), then such functional relationship is called exponential function.

1.2.7 Suggested Readings :

1. B.M. Aggarwal : Mathematics for Business and Economics
2. S.C. Aggarwal and R.K. Rana : Basic Mathematics for Economists
3. D. Bose : Introduction to Mathematical Economics

1.2.8 List of Questions :

1.2.8.1 Short Question

(i) Define logarithmic function.

(ii) Give examples of exponential function.

(iii) State two properties of Logarithmic function.

(iv) Evaluate $\frac{d}{dx}(\log_5 \sqrt{x})$

1.2.8.2 Long Questions

1. Differentiate the following w.r.to x.

(a) $\log \frac{3x+2}{5x+6}$ (b) $\log \frac{\sqrt{x+1}}{x-1}$

2. Find derivative w.r.to x.

(b) $9\sqrt{8x^2+4x+2}$

M.A. (ECONOMICS) PART-I **PAPER-III**
(BASIC QUANTITATIVE METHODS)

LESSON NO. 1.3 **AUTHOR : DR. VIPLA CHOPRA**

APPLICATION OF SIMPLE DERIVATIVES IN ECONOMICS

Structure

1.3.0 Introduction:

In the previous two lesons we have studied the differentiation of simple functions and logarithmic functions. Various rules for finding derivatives have also been explained. In the present lesson we discuss some simple applications of derivatives in Economics. Simple derivatives have wide range of applications in many topics in economics such as price elasticity of demand, price elasticity of supply, marginal revenue, marginal cost, marginal propensity to consume etc. Our aim is to interpret the derivatives with reference to some economic problems.

1. In derivative, we recall some simple results from the previous lessons. We know that

$$\frac{d}{dx}\left(ax^2 + bx + c\right) = 2ax + b$$

and

$$\frac{d}{dx}\left(9x^3 - 8x^2 + 17x + 11\right) = 27x^2 - 16x + 17$$

We note that in the above example, constants c and 11 do not really produce any effet on the derivatives, because the derivative of a constant term is zero. In contrast to the multiplicative constant, which is retained during the differentiation, the additive constant drops out. This fact provides the mathematical explanation of the well known economic prinicple that the fixed cost of a firm, does not affect the marginal cost.

Given a short-run total cost function

$C = 6Q^3 + 14Q^2 + 12Q + 175$

The marginal cost function is that limit of quotient $\frac{\Delta C}{\Delta Q}$ or

$\frac{dC}{dQ} = 18Q^2 + 28Q + 12,$ whereas the fixed cost is represented by the additive constant 175.

Since the latter drops out during the process of deriving the magnitude of the cost obviously cannot affect the marginal cost.

2. As an application of product rule

$$\frac{d}{dx}\left[f(x)g(x)\right] = f(x)\frac{d}{dx}g(x) + g(x)\frac{d}{dx}f(x)$$

1.3.1 Objectives

After studying this lesson you will be able to

* find marginal revenue function if average revenue function is given.
* work out the relationship between average cost and marginal cost.
* determine revenue maximization, profit maximization.
* learn about the effect of taxation and subsidy on Monopoly.
* find out the price elasticity of demand.

1.3.2 Economic Application of Derivatives:

Now, we shall consider the economic application of derivatives in the following cases:

1.3.2.1 Finding Marginal Revenue Function from Average Revenue Function:

If we are given an Average Revenue (AR) Function in specified form, AR = 20 – 3Q, the Marginal Revenue (MR) Function can be found by first finding.

Total Revenue

i.e. TR = Q . AR

TR = Q (20 – 3Q)

Now $MR = \frac{dR}{dQ} = \frac{d}{dQ}\left[Q(20-3Q)\right]$

$= Q.\frac{d}{dQ}(20-3Q) + (20-3Q)\frac{d}{dQ}.Q$

= Q (–3) + (20 – 3Q).1

MR = 20 – 6Q

But if the AR Function is given in the general form AR = f (Q), then R or TR = AR.Q = Q {F(Q)}

and therefore, $MR = \frac{dR}{dQ}$

$\frac{d}{dQ}\{Qf(Q)\}$

$= Q.\frac{d}{dQ}f(Q) + f(Q)\frac{d(Q)}{dQ}$

MR = f(Q) + Q.f' (Q)

The expression f(Q) + Qf' (Q) is a general result. For any demand function P = f(Q) marginal revenue is equal to f(Q) + Qf' (Q)

Example 1:

Calculate Marginal Revenue

P = $(a - bQ)^2$

Show that (f(Q) + Qf'''(Q))=MR

Solution:

Here f(Q) = $(a - bQ)^2$

f' (Q) = –2b (a – bQ)

MR = f(Q) + Qf' (Q)

= $(a - bQ)^2$ – 2bQ (a – bQ)

= (a – bQ) [a – bQ – 2bQ]

= (a – bQ) [a – 3bQ]

Also P = $(a - bQ)^2$

$MR = \frac{dR}{dQ} = \frac{d}{dQ}\left[pQ\right]$

$$\frac{dP}{dQ} = \left[Q(a-bQ)^2\right] = Q \times 2(a-bQ) \times (-b) + (a-bQ)^2 \times 1$$

$= -2bQ\ (a - bQ) + (a - bQ)^2$

$= (a - bQ)\ [-2bQ + a - bQ]$

$= (a - bQ)\ [a - 3bQ]$ .....(ii)

$\therefore$ $MR = f(Q) + Q\ .\ f'(Q)$

Example 2:

The demand curve for a monopolist is Q = 460 – 12P. Find total revenue, average revenue and marginal revenue. At what value of Q, MR = O ?

Solution:

Here demand law is

Q = 460 – 12P

or 12P = 460 – Q

$$f(Q) = P = \frac{460-Q}{12}$$

$$MR = \frac{dTR}{dQ}$$

Total Revenue (TR) = PQ

$$TR = PQ = \frac{460-Q}{12}Q$$

$$MR = \frac{dTR}{dQ}$$

$$\frac{d}{dQ}\left[\left(\frac{460-Q}{12}\right)Q\right]$$

$$= \frac{1}{12}\left[(460-Q)\frac{dQ}{dQ} + Q\frac{d}{dQ}(460-Q)\right]$$

$$= \frac{1}{12}\left[(460-Q).1 + Q(-1)\right]$$

$$= \frac{1}{12}\left[(460-2Q)\right]$$

$$MR = \frac{230 - Q}{6}$$

$$AR = P = \frac{460 - Q}{12}$$

$$MR = 0 \text{ i.e. } \frac{230 - Q}{6} = 0$$

or Q = 230 At Q = 230 MR will be zero

Here $P = \frac{460 - Q}{12}$

$$= \frac{230}{12} = \frac{115}{6}$$

$$P = 19\frac{1}{6}$$

Hence price $P = 19\frac{1}{6}$ when MR = 0

1.3.2.2 Relationship between Marginal Cost and Average Cost Functions:

Given a total cost function C = C(Q), the average cost is $\frac{C(Q)}{Q}$. Therefore, the rate of change of AC with respect to Q can be found by differentiating AC as follows:

$$\frac{d}{dQ}\left(\frac{C(Q)}{Q}\right) = \frac{QC^1(Q) - C(Q)}{Q^2}$$

$$= \frac{C'(Q)}{Q} - \frac{C(Q)}{Q^2}$$

$$= \frac{1}{Q}\left[C'(Q) - \frac{C(Q)}{Q}\right]$$

$$= \frac{1}{Q}[MC - AC]$$

From this follows that, for Q > 0

$$\frac{d}{dQ}\left[\frac{C(Q)}{Q}\right] > 0, C'(Q) > \frac{C(Q)}{Q}$$

The derivative represents the Marginal Cost (MC) Function and $\frac{C(Q)}{Q}$ represents the AC Function.

The Economic meaning of it is as under:

The slope of AC curve will be +ve, zero or -ve if the marginal cost lies above, intersects or lies below the AC curve.

We can intepret it as follows:

Although cost function may assume many different shapes under different circumstances yet, usually under economic limitation we assume average and marginal costs to have U-shapes.

Relationship between AC and MC follows:

The relationship between them is as follows:

To prove that AC is minimum when AC = MC

(i) When AC falls, MC falls more than AC.

(ii) When AC is minimum, AC = MC.

(iii) When AC rises, MC rises more than AC.

AC is minimum, if the first order condition is satisfied that is

$$\frac{d}{dQ}(AC) = 0$$

or $$\frac{d}{dQ}\left(\frac{C}{Q}\right) = 0$$

and $$\frac{d^2}{dQ^2}(AC) > 0$$

or $$\frac{d^2}{dQ^2}\left(\frac{C}{Q}\right) > 0$$

Now $$\frac{d}{dQ}\left(\frac{C}{Q}\right) = \frac{QC'(Q)}{Q^2}$$

$$= \frac{C'}{Q} - \frac{C}{Q^2} = 0 \qquad -C$$

$$=\frac{1}{Q}\left(C'-\frac{C}{Q}\right)=0$$

$$C'=\frac{C}{Q}=0 \text{ or } MC = AC$$

When

$C = Q^3 - 12Q^2 + 60Q$

$= Q\,[Q^2 - 12Q + 60]$

$$AC=\frac{C}{Q}=Q^2-12Q+60$$

When Q = 0, AC = 60

when Q = 1, AC = 1 – 12 + 60 = 49

when Q = 6, AC = 36 – 72 + 60 = 96 – 72 = 24

when Q = 7, AC = 49 – 84 + 60 = 25

So to the left of Q = 6, AC is declining and the MC lies below it, to the right of Q, the opposite is true.

At Q = 6, AC has zero slope and MC and AC have the same value.

$$MC=\frac{dC}{dQ}=3Q^2-24Q+60$$

$= 3\,[Q^2 - 8Q + 20]$

At Q = 6, MC = 3 [36 – 48 + 20] = 3[8]=24

Thus AC = MC at Q = 6.

$\frac{d}{dQ}\left(\frac{C}{Q}\right)$ i.e. the derivative of AC is known as the slope of AC curve.

Example 3:

Given $C = aQ^2 + bQ + d$ find average and marginal cost.

Also verify that slope of AC curve is

$$=\frac{1}{Q}[MC-AC]$$

Solution:

Given $C = aQ^2 + bQ + d$

then $MC=\frac{dC}{dQ}$

$= 2aQ + b$

$$AC = \frac{aQ^2 + bQ + d}{Q}$$

$$\frac{dC}{dQ}$$

$$AC = aQ + b + \frac{d}{Q}$$

Slope of AC curve $= \frac{d}{Q}[AC]$

$$= \frac{d}{Q}\left(aQ + b + \frac{d}{Q}\right)$$

Slope of AC curve $= a - \frac{d}{Q^2}$ .....(i)

Now $\frac{1}{Q}[MC - AC]$

$$= \frac{1}{Q}\left[2aQ + b - \left(aQ + b + \frac{d}{Q}\right)\right]$$

$$= \frac{1}{Q}\left(aQ - \frac{d}{Q}\right) = \left(a - \frac{d}{Q^2}\right) \quad .....(ii)$$

From (i) and (ii)

Slope of AC curve $= \frac{1}{Q}[MC - AC]$

Example 4:

We consider an example to verify these general marginal and average cost relationships:

Consider $C = Q^3 - 3Q^2 + 15Q$

$AC = Q^2 - 3Q + 15$

$MC = 3Q^2 - 6Q + 15$

Consider when AC (Average Cost) declines

i.e.

$$\frac{d}{dQ}\left(Q^2 - 3Q + 15\right) < 0 \quad .....(i)$$

or (i) $2Q - 3 < 0$

but when AC declines; MC < AC

$(3Q^2 - 6Q + 15) < (Q^2 - 3Q + 15)$

$2Q^2 - 3Q < 0$

$Q\ (2Q - 3) < 0$

or $2Q - 3 < 0$ because $Q > 0$ .....(ii)

that is inequality (i) must hold good if MC is below AC; but the same inequality

(i) Suggests that AC is declining.

(ii) Consider when AVC is rising.

$$\frac{d}{dQ}\left[Q^2 - 3Q + 15\right] > 0$$

$2Q - 3 > 0$ .....(iii)

but when AC rises MC should be above AC

$(3Q^2 - 6Q + 15) > (Q^2 - 3Q + 15)$

$2Q^2 - 3Q > 0$

$Q\ (2Q - 3) > 0$

$2Q - 3 > 0$ .....(iv)

(iii) Consider when AVC is minimum

$$\frac{d}{dQ}\left(Q^2 - 3Q + 15\right) > 0$$

$$2Q - 3 > 0 \text{ or } Q > \frac{3}{2}$$

but when AVC is minimum AC = MC.

$Q^2 - 3Q + 15 = 3Q^2 - 6Q + 15$ or $2Q^2 - 3Q = 0$, $Q(2Q-3) = 0$ or $Q = \frac{3}{2}, 0$

1.3.2.3 Maximum Total Revenue:

We can determine the level of output Q where total revenue (TR) will be maximum, since we know that TR is maximum when MR = 0

Example 5:

If the demand function is $P = \sqrt{9 - Q}$ find that at what level of Q, TR will be maximum and what will it be ?

Now TR = PQ = Q $\sqrt{9 - Q}$

$$MR = \frac{dTR}{dQ} = \frac{Q}{2}(9 - Q)^{-1/2}(-1) + (9 - Q)^{-1/2}$$

$$= \frac{-Q}{2\sqrt{9-Q}} + \sqrt{9-Q}$$

$$= \frac{2(9-Q)-Q}{2(9-Q)^{1/2}}$$

$$MR = \frac{18-3Q}{2(9-Q)^{1/2}}$$

For maximum revenue

MR = 0

i.e. 18 – 3Q = 0

or Q = 6

$$\frac{d^2TR}{dQ^2} = \frac{d}{dQ}\left(\frac{18-3Q}{2\sqrt{9-Q}}\right)$$

$$= \frac{2\sqrt{9-Q}(-3)-(18-3Q)(9-Q)^{-1/2}(-1)}{4(9-Q)}$$

$$= \frac{3Q-36}{4(9-Q)^{3/2}}$$

at Q = 6, $\frac{d^2TR}{dQ^2}$ is negative.

Hence TR is maximum

When Q = 6

Maximum total revenue is

$$= \sqrt{9-Q}.Q$$

$$= \sqrt{9-6}.6$$

$$= 6\sqrt{3}$$

Exercise 1

1. Explain the concept of Average Revenue and Marginal Revenue.

Ans. .....

.....

2. Give the demand curve of a monopolist P=25-$\frac{q}{4}$. Find out TR and A.R. and M.R.

3. Explain Average Cost and Marginal Cost.

1.3.2.4 Conditions for Profit Maximisation:

Assuming that we are given the total cost function along with the total revenue function both in terms of output and we are interested in firm's output level which would maximise its total revenue

Let Total Cost TC = C (Q) and Total Revenue TR = ϕ(Q)

Total Profit = TR – TC

$$\pi = \phi(Q) - C(Q)$$

<u>First Order Condition</u> for Maximum Profit $\frac{d\pi}{dQ} = 0$

i.e. $$\frac{d\phi}{dQ} - \frac{dC}{dQ} = 0$$

or $$\frac{d\phi}{dQ} = \frac{dC}{dQ}$$

i.e. MR = MC

<u>Second Order Condition</u>:

$$\frac{d^2\pi}{dQ^2} = \frac{d^2\phi}{dQ^2} - \frac{d^2C}{dQ^2}$$

if $$\frac{d^2\phi}{dQ^2} - \frac{d^2C}{dQ^2} \leq 0$$

at that level of output for which MR = MC, then profit is maximum accordingly.

It would mean that rate of change of MR (slope of MR) should be less than the rate of change of MC (slope of MC) at the profit maximising output

level given by MR = MC.

Example 6:

A plant produces Q tons of steel per week at the total cost of Rs. $\frac{1}{3}Q^3 - Q^2 + 452Q + 50$.

If the market price is fixed at Rs. 500 per ton, show that the plant should produce 8 tons per week.

Solution:

$$TC = \frac{1}{3}Q^3 - Q^2 + 452Q + 50$$

TR = Q.500 = 500Q

π = (Profit) = TR – TC

$$\pi = 500Q - \frac{1}{3}Q^3 + Q^2 - 452Q - 50$$

$$\pi = 48Q - \frac{1}{3}Q^3 + Q^2 - 50$$

For Profit Maximisation

$$\frac{d\pi}{dQ} = 0$$

or $$\frac{d}{dC}\left[48Q - \frac{1}{3}Q^3 + Q^2 - 50\right] = 0$$

$48 - Q^2 + 2Q = 0$

$Q^2 - 2Q - 48 = 0$

$(Q - 8)(Q + 6) = 0$

$Q - 8 = 0$ ($Q = -6$ is not possible).

Now = $(Q - 8)(Q + 6)$

at Q = 8; $\frac{d^2\pi}{dQ^2} = -2Q+2 = -16 + 2 = -14 < 0$

Hence for maximum profit, plant should produce 8 tons per week.

Example 7:

A radio manufacturer produces x sets per week at a total cost of Rs. $\left(\frac{x^2}{25} + 3x + 100\right)$

He is a monopolist and the demand for his market is x = 75 – 3p; where p is the price in rupees per set. Show that maximum net revenue is obtained when about 30 sets are produced per week.

Solution:

$$TC = \frac{x^2}{25} + 3x + 100$$

Demand Law: x = 75 – 3p

or p = 25 – x/3

$$TR = Px = \left(25 - \frac{x}{3}\right)x = 25x - \frac{x^2}{3}$$

Net Revenue (π) = TR – TC

$$\pi = 25x - \frac{x^2}{3} - \frac{x^2}{25} - 3x - 100$$

$$\pi = 22x - \frac{28x^2}{75} - 100$$

$$\frac{d\pi}{dx} = 22 - \frac{56x}{75}$$

For Maximum Net Revenue

$$\frac{d\pi}{dx} = 0$$

$$22 - \frac{56x}{75} = 0$$

$$x = \frac{22 \times 75}{56}$$

$$x = \frac{1650}{56} = 30 \text{ units approx.}$$

$$\frac{d^2\pi}{dx^2} = -\frac{56}{75} < 0$$

Hence net revenue is maximum when the firm produces about 30 sets per week.

1.3.2.5 Effect of Taxation and Subsidy on Monopoly:

Suppose a Monopolist has the following types of total cost and demand

laws:

$$C = \alpha Q^2 + \beta Q + \gamma \qquad (\alpha, \beta, \gamma > 0)$$

$$p = a - bQ \qquad (p, b > 0)$$

Assuming that the government plans to levy a tax 't' per unit quantity upon the commodity produced by the monopolist, problem is : what tax 't' should government impose to get the maximum tax revenue.

As we know that the imposition of sales tax etc. upon a commodity sold reduces the profit and output level and increases the price level of monopolist.

Total tax = Qt.

$$TC = C + tQ = \alpha Q^2 + \beta Q + \gamma + Qt$$

$$MC = 2\alpha Q + \beta + t$$

$$TR = (a - bQ)\ Q = aQ - bQ^2$$

$$MR = a - 2bQ$$

For Maximum Profit

MR = MC

$$\alpha - 2bQ = 2\alpha Q + \beta + t$$

$$Q(2\alpha + 2b) = \alpha - \beta - t$$

or $$Q = \frac{\alpha - \beta - t}{2\alpha + 2b} = \frac{\alpha - \beta - t}{2(\alpha + b)}$$

This Q denotes the level after the tax 't' has been imposed.

Since the govt. had levied tax 't' per unit, then the total tax is t Q.

$$Q = \frac{(\alpha - \beta - t)t}{2b + 2\alpha}$$

For max. tax $$\frac{d}{dt}\left(\frac{(\alpha - \beta - t)t}{2b + 2\alpha}\right) = 0$$

$$\frac{(\alpha - \beta - t)t}{2ab + 2\alpha} = 0 \qquad \text{or} \qquad \alpha - \beta - 2t = 0$$

or $$t = \frac{\alpha - \beta}{2}$$

Now $\frac{d^2}{dt^2}\left(\frac{(\alpha-\beta-t)\,t}{2b+2\alpha}\right)=\frac{d}{dt}\left(\frac{(\alpha-\beta-t)\,t}{2b+2\alpha}\right)$

$=-\frac{2}{2b+2\alpha}$, which is true, α, b are positive.

Hence tax is maximum.

and $t=\frac{\alpha-\beta}{2}$

Note:

It should be noted that subsidy is a negative tax.

Suppose it is the subsidy per unit.

Then TC = – sQ

$=\alpha Q^2+2\beta Q+\gamma-3$

We can now discuss the problem as we have done it earlier.

Example 8:

Under perfect competition, the price of Rs. 6 per unit has been determined. An individual firm has total cost function given by

$\alpha=10+15Q-3Q^2+\frac{1}{3}Q^3$

Show that it is better for the firm to shut down.

Solution:

TC = 10 + 15Q – $3Q^2$ + $\frac{1}{3}Q^3$

TR = 6Q

π (Profit) = TR – TC

π = 6Q – $\left(10+15Q-3Q^2+\frac{1}{3}Q^3\right)$

$\pi=-10-9Q+3Q^2-\frac{1}{3}Q^3$

For max. profit $\frac{d\pi}{dQ}=0$

or – 9 + 6Q – Q^2 = 0

$Q^2 - 6Q + 9 = 0$

$(Q - 3)^2 = 0$ or $Q = 3$

at Q = 3, π is = –10 – 27 + 27 – 9 = –19

The firm has a loss of Rs. 19. If the firm chooses to shut down then its loss will be the fixed cost i.e. 10 $\left(\text{given in TC} = 10 + 15Q - 3Q^2 + \frac{1}{3}Q^3\right)$

Hence the firm finds it better to shut down.

Example 9:

Following are market demand and supply functions

Q = 900 – p

S = 2p

Find the tax rate which maximises the tax yield.

Solution:

Let tax levied be 't' per unit

Tax = tQ

When tax is levied, the supplier will get (p – t) price per unit of quantity supplied. Hence the supply function will take the following form:

S = 2 (p – t)

In equilibrium S = Q

2 (p – t) = 900 – p

2p – 2t = 900 – p, or 3p = 900 + 2t

$$p = 300 + \frac{2}{3}t \quad \ldots\ldots\ldots\ldots\text{(i)}$$

Now substituting this value of p in demand function

Q = 900 – p

$$Q = 900 - \left(300 + \frac{2}{3}t\right)$$

$$Q = 600 - \frac{2}{3}t$$

Total tax = tQ = $\frac{(\alpha - \beta - t)t}{2ab + 2\alpha} = 0$ = $600t - \frac{2}{3}t^2$

For max. tax yield

$$\frac{d}{dt}\left[600t - \frac{2}{3}t^2\right] = 0$$

or $600 - \frac{4}{3}t = 0$

or $\frac{4}{3}t = 600$

or $t = \frac{600 \times 3}{4} = \frac{1800}{4}$

$\therefore$ t = Rs. 450

We have seen earlier that per unit variable tax increases the price, reduces the output and profit of the monopolist. However, a lump sum tax will reduce the monopoly profit, but it will not change his optimum price quantity combination.

Example 10:

Let p = 20 – Q and C = 20 + 4Q and tax rate of Rs. 2 per unit is imposed. Show that the tax reduces the profits and output and increases the prices. But if a lump sum tax of Rs. 14 is imposed. It neither diminishes the output nor decreases the price.

Solution:

Without Tax:

Here p = 20 – Q

C = 20 + 4Q

Profit i.e. π = TR – TC

= (20 – Q) Q – (20 + 4Q)

= $-Q^2 + 16Q - 20$

$$\frac{d\pi}{dQ} = -2Q + 16$$

$$\frac{d^2\pi}{dQ^2} = -2Q < 0$$

$$\frac{d\pi}{dQ} = 0$$

$\therefore$ Q = 8

Here price is 20 – 8 = 12

and profit = –64 + 128 – 20

= Rs. 44

With Tax (Variable):

TR = PQ – tax

$= (20 - Q)\,Q - 2Q$
$= 18Q - Q^2$

$TC = 20 + 4Q$

$\pi = TR - TC$
$= 18Q - Q^2 - (20 + 4Q)$
$= 14Q - Q^2 - 20$

$$\frac{d\pi}{dQ} = 14 - 2Q$$

$$\frac{d^2\pi}{dQ^2} = -2 < 0$$

For max. profit $\frac{d\pi}{dQ} = 0$

or $Q = 7$

Price is $p = 20 - Q = 20 - 7 = 13$

Profit $= 14.7 - 7^2 - 20$

$= 98 - 49 - 20 = 29$

So, profit output have decreased, price has increased.

Lump Sum Tax Levied:

Here lump sum tax = 14

$TR = PQ - 14$

$= 20\,Q - Q^2 - 14$

$TC = 20 + 4Q$

$$\pi = 16Q - Q^2 - 34$$

$$\frac{d\pi}{dQ} = 16 - 2Q$$

$$\frac{d^2\pi}{dQ^2} = -2 < 0$$

Here $Q = 8$ $p = 20 - Q = 20 - 8 = 12$

So price and output remain unchanged.

Exercise 2

1. State the conditions for profit maximisation.

Ans. .....

.....

1.3.2.6 Price Elasticity of Demand:

Price elasticity of demand is defined as the value of the ratio of the relative (or proportionate) change in demand to the relative (or proportionate) change in the price.

If the demand function is Q = f(P) where Q is the quantity demanded and the price is p.

It is assumed that Q varies inversely as P.

Thus, according to definition, price elasticity of demand (i.e. e_d) is expressed as

$$e_d = \frac{P}{Q} \underset{\Delta p \to 0}{Lt} \frac{f(p+\Delta p) - f(p)}{\Delta p}$$

$$= \frac{P}{f(P)} \cdot \frac{dQ}{dP}$$

Since the normal demand curve is monotonic decreasing function, the elasticity of demand will be negative at all prices. As a convention, elasticity of demand is taken as a positive quantity, so negative sign is attached to $\frac{P}{Q} \cdot \frac{dQ}{dP}$

i.e. $$e_d = \frac{P}{Q} \cdot \frac{dQ}{dP} = -\frac{dQ/Q}{dP/P} = -\frac{dQ/dP}{Q/P}$$

$$e_d = \frac{\text{Marginal Demand Function}}{\text{Average Demand Function}}$$

We write e_d in the form $|e_d|$ which means that we only consider the absolute value of e_d irrespective of its sign : hence forth we shall always take absolute (positive) value of the elasticity.

1. If $|e_d| > 1$, at a particular price, we say that demand is elastic at that price.
2. If $|e_d| < 1$, demand is inelastic at that price.
3. If $|e_d| = 1$, we say that demand has unit elasticity at the given price.

We can also write elasticity of demand as under:

$$e_d = \frac{-d(\log Q)}{d(\log P)}$$

Example 11:

A demand function is given by $Q = bP^{-n}$. Calculate price elasticity of demand. What happens when n = 1 ?

Solution:

$Q = bp^{-n}$

$$\frac{dQ}{dP} = -bn\,P^{-n-1}$$

$$e_d = \frac{-dQ/dP}{Q/P} = \frac{-bnP^{-n-1}}{Q/P}$$

$$e_d = \frac{-bnP^{-n-1}}{bP^{-n-1}} = n$$

So elasticity is equal to n at all level of prices.

when n = 1 $e_d = 1$

The curve $Q = bP^{-1}$ is called the constant outlay curve and price elasticity of demand is equal to unity at every point. Such a demand curve is of the shape of a rectangual hyperbola.

Relation between Average Revenue, Marginal Revenue and Elasticity of Demand:

We shall prove that

$$e_d = \frac{AR}{AR - MR}$$

We know that $MR = \frac{d}{dQ}(TR)$

$$MR = P + Q\frac{dP}{dQ}$$

AR = P

$$\therefore \quad MR = AR + Q\frac{dP}{dQ} \qquad \ldots\ldots\ldots\ldots\text{(i)}$$

Also $e_d = \frac{-P}{Q}.\frac{dQ}{dP}$

or $$Q\frac{dP}{dQ} = \frac{-P}{e_d} = \frac{-AR}{e_d}$$

Put in (i)

$$\therefore \quad MR = AR - \frac{AR}{e_d}$$

$$\frac{AR}{e_d} = AR - MR$$

or $$e_d = \frac{AR}{AR - MR}$$

Example 12:

Show that $e_d = \frac{AR}{AR - MR}$

P = a – bQ. Show that e_d = 1 when MR = 0

Solution :

Here P = a – bQ

$\therefore$ AR = a – bQ

$$MR = \frac{d}{dQ}(TR) = \frac{d}{dQ}(P.Q)$$

$$= \frac{d}{dQ}([a - bQ]Q)$$

MR = a – 2bQ

Now $e_d = \frac{-P}{Q} \cdot \frac{dQ}{dP}$

$$e_d = -\frac{P}{Q} \cdot \frac{dQ}{dP} = -\frac{dQ/Q}{dP/P} = -\frac{dQ/dP}{Q/P}$$

$$\therefore \quad e_d = \frac{-P}{Q} \cdot \frac{dQ}{dP}$$

$$e_d = \frac{-(a - bQ)}{Q}\left(\frac{-1}{b}\right)$$

$$e_d = \frac{a - bQ}{bQ}$$

Also $\frac{AR}{AR-MR}=\frac{a-bQ}{(a-bQ)-(a-2bQ)}$

$=\frac{a-bQ}{bQ}$

Hence $e_d=\frac{AR}{AR-MR}$

When MR = 0 i.e. a – 2bQ = 0 or $Q=\frac{a}{2b}$ $\therefore$ a = 2bQ

Also $e_d=\frac{a-bQ}{(a-bQ)-(a-2bQ)}=\frac{2bQ-bQ}{bQ}=\frac{bQ}{bQ}=1$

Example 13:

Given the demand function as P = 20 – 2x, find the price elasticity of demand at p = 4 and p = 8.

Solution:

Given P = 20 – 2x

$\frac{dP}{dx}=-2$ $\therefore \frac{dx}{dP}=-\frac{1}{2}$

(i) At p = 4, 4 = 20 – 2x $\Rightarrow$ 2x = 16 or x = 8

$e_d=-\frac{P}{x}.\frac{dx}{dP}=\left(-\frac{4}{8}\right)\left(-\frac{1}{2}\right)=\frac{1}{4}=0.25$

(ii) At p = 8, 8 = 20 – 2x $\Rightarrow$ 2x = 12 or x = 6

$e_d=-\frac{P}{x}.\frac{dx}{dP}=\left(-\frac{8}{6}\right)\left(-\frac{1}{2}\right)=\frac{8}{12}=\frac{2}{3}=0.67$

1.3.3 Summary:

In this lesson you have studied the economic application of derivatives. Derivative which is concerned with determining the rate of change of a given function due to a unit change in the independent variable. Many economic problems relating to margianl revenue (MR), marginal cost (MC), marginal propensity to consume (MPC), profit maximisation revenue and cost minimisation have been explained in this lesson. The concepts of price elasticity of demand and price elasticity of supply using derivatives.

1.3.4 Key Words:

Marginal Revenue : Rate of change in total revenue with respect to output.

Marginal Cost : Rate of change in total cost with respect to output.

Price elasticity of demand : Elasticity of demand is the relative change in demand in respect to a relative change.

1.3.5 Suggested Readings :

1. S.C. Aggarwal and R.K. Rana : Basic Mathematics for Economists.
2. B.C. Mehta and G.M.K. Madani : Mathematics for Economists.
3. C.S. Aggarwal and R.C. Joshi : Mathematics for Students of Economics.

1.3.6 List of Questions :

1.3.6.1 Short Questions

(i) Define price elasticity of demand mathematically.

(ii) State conditiosn of profit maximisation.

(iii) Define the following concepts mathematically.
(i) Marginal Revenue, (ii) MC

(iv) State the relation between Average Revenue, Margianl Revenue and Elasticity of Demand.

1.3.6.2 Long Questions

(i) Given the demand function: P = 50 – 3q,

Show that $E = \frac{AR}{AR - MR}$ at P = 5

(ii) Determine price elasticity of demand and marginal revenue if $q = 30 - 4P - P^2$ where q is the quantity demanded and P is the price and P = 3

(iii) The total cost function of a firm is given as:

$\pi = 3 + 2q + 5q^2$

Find AC and MC and hence show that

slope of $AC = \frac{1}{q}$ (MC – AC)

LESSON NO. 1.4 **AUTHOR : DR. VIPLA CHOPRA**

PARTIAL DERIVATIVES AND EULER'S THEOREM

1.4.0 Introduction:

In Lessons 1-3, we have considered problems relating to function having one independent variable. Now, we will extend the idea of differentiation to cover functions involving two or more independent variables.

Consider U = f (x, y) ..... (i), where x and y are independent variables and U is dependent variable.

In Economics, we can cite many such examples. For instance, total utility is a function of the quantity of the commodities x and y consumed. Demand of tea is a function of price of tea and price of coffee. Production of wheat depends on land and

labour. Here in these above examples, one of the variable can be assigned a fixed value and the other variable is varied. In this way, two functions can be obtained one U as a function of x keeping y constant while the other U as a function of y keeping x constant. The derivatives so obtained are called partial derivatives of the function.

$U = f(x, y)$ .....(i)

If the variable x undergoes change by Δx while y remains constant, there will be a corresponding change in U, say ΔU, then

$U + \Delta U = f(x + \Delta x, y)$ .....(ii)

Subtract (i) from (ii), we get

or $\Delta U = f(x + \Delta x, y) - f(x, y)$

Divide both sides by Δx, we get

or $$\frac{\Delta U}{\Delta x} = \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}$$

Now take limits as Δx tends to zero,

$$\underset{\Delta x \to 0}{Lt} \frac{\Delta U}{\Delta x} = \underset{\Delta x \to 0}{Lt} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x} \text{ or } \frac{\partial \mu}{\partial x} = \underset{\Delta x \to 0}{Lt} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}$$

This specific derivative is known as "Partial Derivative" of U with respect to x. Similarly, the partial derivatives of function (i) with respect to y is

$$\underset{\Delta y \to 0}{Lt} \frac{\Delta U}{\Delta y} = \underset{\Delta y \to 0}{Lt} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y} \quad or \frac{\partial \mu}{\partial y} = \underset{\Delta y \to 0}{Lt} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y}$$

1.4.1 Objectives

After studying this lesson, you should know the

* basic concept of partial derivatives.
* difference between partial differentiation and simple differentiation.
* meaning of Homogeneous function.
* types of homogeneous function.
* Euler's Theorem.

1.4.2 Partial Derivatives:

In this section, we will discuss definition of partial derivatives and techniques of partial differentiation.

1.4.2.1 Definition:

The partial derivative of U = f (x, y) with respect to x at point (x, y) is defined as $\underset{\Delta x \to 0}{Lt} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}$, if it exists finitely.

We call it "Partial" derivative to indicate that y (or all other variables in case there are more than two variables) in the function has been held constant. This renders the function of two variables as the function of a single variable. In place of letter 'd', we use symbol '∂' (curly) and write partial derivative of U with respect to x at (x, y) as

$\frac{\partial U}{\partial x}\left(\text{not as}\frac{dU}{dx}\right)$ or f_x or $f_x^1(x, y)$.

Similarly, we can have the partial derivative of U with respect to y as:

$\frac{\partial U}{\partial y}$ or f_y or $f_y^1(x, y)$

1.4.2.2 Technique of Partial Differentiation:

The process of taking partial derivative is called partial differentiation. It differs from simple differentiation primarily in that we hold and treat all the independent variables constant except the one which is assumed to vary.

1.4.2.3 First Order Partial Differentiation:

If U = f (x, y) where x and y are independent variables and U is a dependent variable. Then first order partial derivatives of this function are

$\frac{\partial U}{\partial x}$ or f_x and $\frac{\partial U}{\partial y}$ or f_y

Example 1:

If $Z = 4x^2 + 6xy + 2y^2$ .....(i)

We have first to find partial derivative of (i) w.r.t. x

$\frac{\partial z}{\partial x} = 8x + 6y$ (y is held constant).

Similarly, differentiating (i) w.r.t. y, we get

$\frac{\partial z}{\partial y} = 6x + 4y$ (x is held constant).

Example 2:

Let $U = \frac{1}{\sqrt{x^2 + y^2}}$

Differentiating u w.r.t. x, we get

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left(x^2 + y^2\right)^{-1/2} = \frac{-1}{2}\left[x^2 + y^2\right]^{-3/2} \frac{\partial}{\partial x}\left(x^2 + y^2\right)$$

$$= \frac{-1}{2}\left(x^2 + y^2\right)^{-3/2}(2x) = \frac{-x}{\left(x^2 + y^2\right)^{3/2}}$$

Also differentiating w.r.t. y, we get

$$\frac{\partial u}{\partial y} = \frac{\partial}{\delta y}\left(x^2 + y^2\right)^{-1/2}$$

$$\frac{\partial u}{\partial x} = -\frac{1}{2}\left(x^2 + y^2\right)^{-3/2}(2y) = \frac{-y}{\left(x^2 + y^2\right)^{3/2}}$$

Exercise 1:

1. Find Partial derivatives (first order) of the following:

   (i) $u = \sqrt{xy}$ (ii) $(x^3 + y^3)^2$ (iii) $U = e^{x2} - y^2$

2. If $x^3 - 3x^2y + 3xy^2 - y^3$, show that

   $$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial ty} = 3u$$

3. If $u = \log \sqrt{x^2 + y^2}$

   Show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial ty} = 1$

4. If $U = x^2 \log \left(\frac{y}{x}\right)$

   Prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial ty} = 2u$

1.4.2.4 Second Order Partial Differentiation:

If U = f (x, y) be a function involving two independent variables and one dependent variable, we can find the second or higher order partial derivatives. The process of derivation can be repeated till the partial derivative happens to be the function of any of the independent variables in the original function.

Now U = f (x, y)

$\frac{\partial u}{\partial x}$ = First order partial derivative of U w.r.t. x, keeping y constant.

$\frac{\partial u}{\partial y}$ = First order partial derivative of U w.r.t. y, keeping x constant.

1. The partial derivatives $\frac{\partial u}{\partial x}$ w.r.t. x is $\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)$ and is denoted by $\frac{\partial^2 u}{\partial x^2}$ or f_{xx}. In

other words $f_{xx} = \left(\frac{\partial^2 u}{\partial x^2}\right)$ is the second order partial derivatives obtained by

partial derivation, first w.r.t. x and then again w.r.t. x

2. The partial derivative of $\frac{\partial u}{\partial x}$ w.r.t. y is $\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)$ or $\frac{\partial^2 u}{\partial x\, \partial y}$ or f_{xy}. Thus f_{xy} is the

second order partial derivative.

3. Similarly, if $\frac{\partial u}{\partial y}$ happens to be the function of both x and y, it could be

partially differentiated w.r.t. x and y, so that when it is partially differentiated

w.r.t. x, we have $\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)$ or $\frac{\partial^2 u}{\partial y\, \partial x}$ or f_{yx}.

4. We can also have $\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)$ or $\frac{\partial^2 u}{\partial y^2}$ or f_{yy}. Thus $f_{xx} f_{yx} f_{xy}$ and f_{yy} are four second

order partial derivatives of the function.

1.4.2.5 Cross Partial Derivatives:

Second order partial derivatives f_{xy} and f_{yx} are also known as cross partial derivatives.

The cross partial derivatives $f_{yx} = \left(\frac{\partial^2 u}{\partial x\, \partial y}\right)$ and $f_{xy} = \left(\frac{\partial^2 u}{\partial y\, \partial x}\right)$ are different only

in order in which u has been differentiated partially. However, under certain conditions (relating to continuity of the function) the cross partial derivatives are identical in value, i.e.

$$\frac{\delta^2 u}{\delta y\, \delta x} = \frac{\delta^2 u}{\delta x\, \delta y} \quad \text{or} \quad f_{xy} = f_{yx}$$

Thus the order of the partial derivation does not make any difference in the process of the partial differentiation.

Example 3:

Consider $u = 3x^2 + 9xy - 2y^2$

Now $f_x = \frac{\delta u}{\delta x} = \frac{\delta}{\delta x}\left(3x^2 + 9xy - 2y^2\right)$

$f_x = 6x + {}^{y}9y$

$f_y = \frac{\delta u}{\delta y} = \frac{\delta}{\delta t}\left(3x^2 + 9xy - 2y^2\right)$

$fy = 9x - 4y$

f_x and f_y are first order partial derivatives of the function. Next, we have to find second order partial derivatives i.e. f_{xx}, f_{xy}, f_{yx} and f_{yy}

$$f_{xx} = \frac{\delta^2 u}{\delta x^2} = \frac{\delta}{\delta x}\left(\frac{\delta u}{\delta x}\right) = \frac{\delta}{\delta x}(6x + 9y) = 6$$

$$f_{xy} = \frac{\delta^2 u}{\delta y\, \delta x} = \frac{\delta}{\delta y}\left(\frac{\delta u}{\delta x}\right) = \frac{\delta}{\delta y}(6x + 9y) = 9$$

Now $f_y = 9x - 4y$

$$f_{yy} = \frac{\delta^2 u}{\delta y^2} = \frac{\delta}{\delta x}\left(\frac{\delta u}{\delta y}\right) = \frac{\delta}{\delta y}(9x - 4y) = -4$$

$$f_{yx} = \frac{\delta^2 u}{\delta x\, \delta y} = \frac{\delta}{\delta x}\left(\frac{\delta u}{\delta y}\right) = \frac{\delta}{\delta x}(9x - 4y) = 9$$

Example 4:

Let $Z = x^3 e^{2y}$

Take partial derivative of Z w.r.t. x

$$\therefore \frac{\partial z}{\partial x} = e^{2y}.3x^2$$

$$f_{xx} = \frac{\delta}{\delta x}\left(\frac{\delta z}{\delta x}\right) = \frac{\delta}{\delta x}\left(3x^2e^{2y}\right) = 6xe^{2y}$$

Take partial derivative of z w.r.t. y, we get,

$$f_y = \frac{\delta z}{\delta y} = 2x^3e^{2y}$$

$f_{yx} = 6x^2e^{2y} + 0.\ (e^{2y}) = 6x^2e^{2y}$
$f_{xy} = 2x^3\ (0) + 6x^2e^{2y} = 6x^2e^{2y}$
$f_{yy} = 3x^3e^{2y}\ (2) + 0.\ (e^{2y}) = 4x^3e^{2y}$

Example 6:

If $Z = \log (x^2 + y^2)$ show that $\frac{\delta^2 z}{\delta x^2} + \frac{\delta^2 z}{\delta y^2} = 0$

Now $Z = \log (x^2 + y^2)$

$$\therefore\ \frac{\partial z}{\partial x} = \frac{1}{\left(x^2 + y^2\right)}.2x = \frac{2x}{x^2 + y^2}$$

$$\frac{\delta}{\delta x}\left(\frac{\delta z}{\delta x}\right) = \frac{(x^2 + y^2)2 - 2x(2x)}{(x^2 + y^2)^2} = \frac{2x^2 + 2y^2 - 4x^2}{(x^2 + y^2)^2}$$

$$\frac{\delta^2 z}{\delta x^2} = \frac{2y^2 - 2x^2}{(x^2 + y^2)^2} = \frac{2(y^2 - x^2)}{(x^2 + y^2)^2}$$

$$\frac{\delta z}{\delta y} = \frac{2y}{x^2 + y^2}$$

$$\frac{\delta^2 z}{\delta y^2} = \frac{\left(x^2 + y^2\right)(2) - 2y\,(2y)}{\left(x^2 + y^2\right)^2} = \frac{2\left(x^2 - y^2\right)}{\left(x^2 + y^2\right)^2}$$

L.H.S. $\frac{\delta^2 z}{\delta x^2} + \frac{\delta^2 z}{\delta y^2} = \frac{2\left(y^2 - x^2\right)}{\left(x^2 + y^2\right)^2} + \frac{2\left(x^2 - y^2\right)}{\left(x^2 + y^2\right)} = 0 = \text{R.H.S.}$

Exercise 2:

1. Find first and second order partial derivatives of the following:
   (i) $u = e^{x+y}$
   (ii) $u = x^2y + y^2x$
   (iii) $u = (x^2 - y2)^{4/3}$
   (iv) $z = \log (x^2 + y^2)^{2/3}$
2. If $z = 2\ (ax + by)^2 - (x^2 + y^2)$

show that $\frac{\delta^2 u}{\delta x^2} + \frac{\delta^2 u}{\delta y^2} = \left(a^2 + b^2\right) - 4$

3. If $a^2x^2 + b^2y^2 = c^2u$, show that

$$\frac{b^2\delta^2 u}{\delta x^2} + \frac{a^2\delta^2 u}{\delta y^2} = \frac{4a^2 + b^2}{c^2}$$

4. Find f_{xx} and f_{yy} of the following:
   (i) $f(x, y) = (2x + 5y)\, e^y$
   (ii) $f(x, y) = x + ye^{-x}$

1.4.3 Homogenous Function:

Now in this section, we will first give definition of homogeneous function. Types and properties of homogeneous function will follow afterwards.

1.4.3.1 Definition:

If $u = f(x, y)$ be a function of two variables then this function is said to be homogeneous function of degree n (or of order in) if the following relationship holds.

$f(tx, ty) = t^n f(x, y)$, $t > 0$ (any positive real number) .....(i)

Thus a function is homogeneous of degree n if each of its factors is changed in the same proportion (t), the new function is t^n times the original function. The value n explains the degree of a homogeneous function.

Consider a production function involving two variables namely labour (L) and capital (C) so that

$q = f(L, C)$

When the inputs L and C are changed in some proportion (t) the output will be changed by some power of the proportionate change (t^n). Here n explains operation of returns to scale.

1.4.3.2 Types of Homogeneous Function:

Homogeneous function can be of the following types:

1. Homogeneous function of degree less than one.
2. Homogeneous function of degree zero.
3. Homogeneous function of degree one.
4. Homogeneous function of degree greater than one.

1.4.4 Properties of Linear Homogeneous Function:

We have seen under section 4.2 the definition of linear homogeneous function. These properties will also be extended to homogeneous function of degree n.

If $\mu = f(x, y)$ is assumed to be a linearly homogeneous function, then following three properties hold good for all values of independent variables x and y.

1.4.4.1 Property I:

This property states that a linear homogeneous function $\mu = f(x, y)$ can be expressed as

$f(x, y) = x\ \phi\ (y/x)$ or $y\Psi'(x/y)$

where ϕ and ψ are some functions of a single variable.

Proof:

Since the given function is linearly homogeneous

$f(tx, ty) = tf(x, y)$ .....(i)

As t can be any value; Put $t = \frac{1}{x}$

$$f\left(1, \frac{y}{x}\right) = \frac{1}{x} f(x, y)$$

i.e. $$f(x, y) = x.f\left(1, \frac{y}{x}\right)$$

Similarly if $t = \frac{1}{y}$

$$f\left(\frac{x}{y}, 1\right) = \frac{1}{y} f(x, y)$$

$\therefore$ $$f(x, y) = y\phi(x/y)$$

Example 8:

$f(x, y) = x^3y - xy^3$

$$= x^4\left(\frac{x^3y}{x^4} - \frac{xy^3}{x^4}\right) = x^4\left(\frac{y}{x} - \frac{y^3}{x^3}\right)$$

$$= x^4\phi\left(\frac{y}{x}\right) \text{ where } \phi\left(\frac{y}{x}\right) = \frac{y}{x} - \frac{y^3}{x^3}$$

Extension I:

This property of linearly homogeneous production function can be extended to homogeneous function of any degree i.e. if f (x, y) is a homogeneous function of degree n, it can be expressed as $f(x, y) = x^n\phi\left(\frac{y}{x}\right)$

1.4.4.2 Property II:

The partial derivatives of the dependent variable of the given linearly homogeneous function i.e. $\left(\frac{\delta\mu}{\delta x} \text{ and } \frac{\delta\mu}{\delta y}\right)$ are functions of the ratio of x to y.

Proof:

From Property I, we have

$$\mu = x\phi\left(\frac{y}{x}\right)$$

Take partial derivative of μ w.r. to x.

$$\frac{\partial\mu}{\partial x} = \phi\left(\frac{y}{x}\right) + x\phi'\left(\frac{y}{x}\right).\frac{\partial}{\partial x}\left(\frac{y}{x}\right) \quad \text{(By product rule)}$$

$$\therefore \quad \frac{\partial\mu}{\partial x} = \phi\left(\frac{y}{x}\right) - \frac{y}{x^2}\phi'\left(\frac{y}{x}\right)x = \phi\left(\frac{y}{x}\right) - \frac{y}{x}\phi'\left(y/x\right)$$

Thus partial derivative of μ with respect to x is a function of ratio between the independent variables

Also since $u = x\phi\left(\frac{y}{x}\right)$

Take partial derivative w.r. to y, we get

$$\frac{\partial\mu}{\partial y} = x\phi'\left(\frac{y}{x}\right)\frac{\partial}{\partial y}\left(\frac{y}{x}\right) = \phi'\left(\frac{y}{x}\right)$$

i.e. $\frac{\partial\mu}{\partial y}$ is also function of ratio between the independent variables.

Extension II:

Let $\mu = f(x, y)$ be a homogeneous function of degree n, then the first partial derivative of μ is homogeneous of degree n – 1, provided the derivative exists, the second partial derivative is the homogeneous function of degree (n – 2) and so on, so that K^{th} partial derivative is homogeneous function of degree (n – k).

Example 9:

Consider the function $f(x, \Psi) = x^2\Psi^2 + x\Psi^3$

$$f(tx, t\psi) = (tx)^2.(t\psi)^2 + tx.(t\psi)^3$$

$$= t^4x^2\psi^2 + tx.t^3.\psi^3$$

$$= t^4x^2\psi^2 + t^4x\psi^3 = t^4\left(x^2\psi^2 + x\psi^3\right)$$

$\therefore$ $f(x, \psi)$ is a homogeneous function of degree 4.

Now $fx = 2x\psi^2 + \psi^3, f\psi = 2x^2\psi + 3x\psi^2$

$f_{xx} = 2\psi^2, f\psi\psi = 2x^2 + 6x\psi$

$\therefore$ Second partial derivatives are homogeneous function of degree 2.

Exercise 2:

1. Express $f(x, \psi) = x^2\psi^2 - x\psi^2$ as $x\phi\left(\frac{\psi}{x}\right)$

2. Express the following functions in the form

$\mu = \phi\left(\frac{y}{x}\right)$ and $\mu = x^0\phi\left(\frac{y}{x}, \frac{z}{x}\right)$ respectively.

(a) $\mu = f(x, y) = \frac{x^2 + y^2}{x + y}$

(b) $\mu = f(x, y, z) = \frac{x^2}{yz} + \frac{y^2}{xz}$

3. Show that partial derivative of the function $\frac{x^3 + y^3}{x - y}$ are homogeneous of a degree which is less by one than that of the original function.

1.4.4.4 Property III:

The linearly homogeneous functions satisfy the third property known as "Euler's Theorem". This theorem states that value of homogeneous function can always be written as a sum of terms, each of these being the product of the first partial derivative and the corresponding independent variable.

Euler's Theorem:

Euler's Theorem is a special relationship obtained from homogeneous functions and is well known to economists in connection with marginal productivity theory, usually under name of the adding up theorem.

Thus if $\mu = f(x, y)$ is a linear homogeneous function then Euler's theorem states that:

$$x\frac{\partial\mu}{\partial x}+y\frac{\partial\mu}{\partial y}=\mu$$

Similarly if $Z=\phi(x,y)$ is a homogeneous function of degree n, then Euler's theorem states that

$$x\frac{\partial z}{\partial x}+Y\frac{\partial z}{\partial y}=nu$$

Proof:

If Z is homogeneous function of degree n, then

$$z=x^n\phi\left(\frac{y}{x}\right) \qquad \ldots\ldots\ldots\ldots(i)$$

$$\frac{\partial z}{\partial x}=nx^{n-1}\phi\left(\frac{y}{x}\right)+x^n\phi^1\left(\frac{y}{x}\right)\frac{\partial}{\partial x}\left(\frac{y}{x}\right)$$

$$=nx^{n-1}\phi\left(\frac{y}{x}\right)-x^{n-2}y.\phi^1\left(\frac{y}{x}\right)$$

$$\therefore \qquad x\frac{\partial z}{\partial x}=nx^n\phi\left(\frac{y}{x}\right)-x^{n-1}\phi^1\left(\frac{y}{x}\right)y \quad \ldots\ldots\ldots\ldots\ldots(ii)$$

Also $\frac{\partial z}{\partial y}=x^n\phi^1\left(\frac{y}{x}\right)\frac{1}{x}=x^{n-1}\phi^1\left(\frac{y}{x}\right)$

$$\therefore \qquad y\frac{\partial z}{\partial y}=x^{n-1}y.\phi^1\left(\frac{y}{x}\right) \qquad \ldots\ldots\ldots\ldots(iii)$$

Adding (ii) and (iii), we have

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=nx^n\phi\left(\frac{y}{x}\right)-x^{n-1}y.\phi^1\left(\frac{y}{x}\right)+x^{n-1}y.\phi^1\left(\frac{y}{x}\right)$$

$$=nx^n\phi\left(\frac{y}{x}\right)=nz$$

Cor. I:

In particular, if Z is a linear homogenous function, then n = 1 and

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=1.z=z$$

Cor. II:

In particular, if Z is homogeneous function of degree 2, then n = 2 and

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=2z$$

Example 10:

Verify Euler's theorem for $\frac{x+y}{x^2+y^2}$

Since $\mu=\frac{x+y}{x^2+y^2}$ is a homogeneous function of degree –1.

∴ Euler's Theorem states that

$$x\frac{\partial \mu}{\partial x}+y\frac{\partial \mu}{\partial y}=-1.\mu=-\mu$$

$$\frac{\partial \mu}{\partial x}=\frac{\left(x^2+y^2\right)-\left(x+y\right)^{2x}}{\left(x^2+y^2\right)^2}=\frac{-x^2+y^2-2xy}{\left(x^2+y^2\right)^2}$$

$$x\frac{\partial \mu}{\partial x}=\frac{xy^2-x^3-2x^2y}{(x^2+y^2)^2} \quad \text{.....(i)}$$

$$\frac{\partial \mu}{\partial y}=\frac{\left(x^2+y^2\right)(1)-\left(x+y\right)(2y)}{\left(x^2+y^2\right)^2}=\frac{x^2-y^2-2xy}{\left(x^2+y^2\right)^2} \quad y\frac{\partial \mu}{\partial x}=\frac{yx^2-y^3-2xy^2}{\left(x^2+y^2\right)^2}$$

.....(ii)

Add (i) and (ii) we get,

$$x\frac{\partial \mu}{\partial x}+y\frac{\partial \mu}{\partial y}=\frac{xy^2-x^3-2x^2y+yx^2-y^3-2xy^2}{\left(x^2+y^2\right)^2}=\frac{-xy^2-x^3-x^2y-y^3}{\left(x^2+y^2\right)^2}$$

$$=\frac{-(xy^2+x^3+x^2y+y^3)}{(x^2+y^2)^2}=\frac{(x+y)(x^2+y^2)}{(X^2+y^2)^2}$$

$$= \frac{-(x+y)}{(x^2+y^2)} = -\mu$$ which verified the Euler's theorem.

Extension III:

Euler's Theorem can be extended for any number of independent variables. If U = f (x_1, x_2, x_3 ..... x_n) is a homogeneous function of degree h, then the theorem holds good if:

$$x_1 \frac{\partial \mu}{\partial x_1} + x_2 \frac{\partial \mu}{\partial x_2} + x_3 \frac{\partial \mu}{\partial x_3} + ... + x_n \frac{\partial \mu}{\partial x_n} = h\mu$$

Euler's Theorem occupies an important place in economics especially in theory of distribution. It is based on homogeneous function as it enables the producer to decide whether it is profitable or not to pay the factors according to their marginal productivity.

Example 11:

Given Q = $Ak^{-}L^{\beta}$

Euler's Theorem is verified if:

$$K\frac{\partial Q}{\partial K} + L\frac{\partial Q}{\partial L} = (\alpha+\beta)Q \qquad .....(i)$$

Differentiate the given function w.r. to K and L:

$$\frac{\partial Q}{\partial K} = AL^{\beta}\frac{\partial}{\partial K}(K^{\alpha}) = AL^{\beta}\alpha K^{\alpha-1}$$

and $$\frac{\partial Q}{\partial L} = AK^{\alpha}\frac{\partial}{\partial L}(L^{\beta}) = AK^{\alpha}\beta L^{\beta-1}$$

Substituting the values in equation (i), we get

$$K(AL^{\beta}\alpha K^{\alpha-1}) + L(AK^{\alpha}\beta L^{\beta-1}) = (\alpha+\beta)Q$$

L.H.S. = $AL^{\beta}\alpha K^{\alpha} + AK^{\alpha}\beta L^{\beta} = A\alpha K^{\alpha}L^{\beta}(\alpha+\beta)$

R.H.S. = $(\alpha+\beta)Q$

L.H.S. = R.H.S.

Hence the proof.

Example 12:

Verify the Euler's Theorem for the function z = $8x^{0.7}y^{0.3}$

Solution:

Let f (x, y) = $8x^{0.7}y^{0.3}$

f(tx1ty)= $f(tx_1ty) = 8(tx)^{0.7}(ty)^{0.3}$

$= t^{0.7} . t^{0.3} (8x^{0.7} . y^{0.3})$

$f(tx,ty) = t' . f(x.y)$

The function is homogeneous of degree 1.

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 1$$

$z = 8x^{0.7}y^{0.3}$

y

$$\frac{\partial z}{\partial y} = 8(0.7)x^{0.7-1}y^{0.3} \Rightarrow x\frac{\partial z}{\partial x} = 8(0.7)x^{0.7}y^{0.3} \qquad \ldots\ldots\ldots\ldots(i)$$

To verify the Euler's Theorem, we have to prove that

$$\frac{\partial z}{\partial y} = 8x^{0.7}(0.3)y^{0.3-1} \Rightarrow y\frac{\partial z}{\partial y} = 8x^{0.7}(0.3)y^{0.3} \qquad \ldots\ldots\ldots\ldots(ii)$$

Adding (i) and (ii), we have

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 8(0.7)x^{0.7}y^{0.3} + 8x^{0.7}(0.3)y^{0.3}$$

$= 8x^{0.7}y^{0.3} = z$

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = z$$

Hence the theorem is proved.

1.4.5 Summary

The objective of this lesson is to provide with some exposure to the concept of partial derivatives and how this technique of partial differentiation differs from simple differentiation. The technique of obtaining partial derivatives has been explained with illustrations. This has been followed by giving difination of Homogeneous Function, types of homogeneous functions and its properties. Finally number of examples have been given to verify Euler's Theorem.

1.4.6 Key Words

Partial derivative : The derivative of a function of two or more variables w.r.t. one of the independent variables by keeping all other variables unchanged, i.e., constant is called the partial derivative of the function w.r.t. variable :

Linear Homogenous Functions : Homogeneous functions of degree 1 are

called linear homogeneous functions.

1.4.7 Suggested Readings

1.	B.M. Aggarwal	:	Mathematics for Business and Economics.
2.	S.C.Aggarwal and R.K. Rana	:	Basic Mathematics for Economists.
3.	Bhardwaj and Sabharwal	:	Mathematics for Students of Economics.

1.4.8 List of Questions :

1.4.8.1 Short Questions

(i) Explain the concept of partial derivative.

(ii) Define Homogeneous Function.

(iii) State one property of Homogeneous Function.

(iv) State Euler's Theorem.

1.4.8.2 Long Questions

1. State Euler's Theorem. Compute the degree of Homogeneity and verify Euler's theorem for the production function $x = f(1, k) = (al^4 + bk^4)^{1/2}$

2. Let $f(x, y) = x^4 + x^2y^2 + x^4$ show that $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$

3. If $f(x, y) = x^2 - 3xy + y^2$, then prove that

$$\frac{\partial f}{\partial x} - \frac{\partial f}{\partial y} = (x - y)\left[\frac{\partial^2 f}{\partial x^2} - \frac{\partial^2 f}{\partial x \partial y}\right]$$

4. Verify Euler's Theorem for the function $Z = \frac{x^{1/4} + y^{1/4}}{x^{1/5} + y^{1/5}}$

M.A. (ECONOMICS) PART-I **PAPER-III**
(BASIC QUANTITATIVE METHODS)

LESSON NO. 1.5 **AUTHOR : DR. VIPLA CHOPRA**

INTEGRATION

Structue:

1.5.1 Introduction:

In differential calculus, we have studied the method of finding out the derivative (differential co-efficient) of a given function with respect to its variable. In other words, we find rate of change in the function. Integration is defined as the reverse (or inverse) process of differentiation. In integration, we are given the rate of change in the function and we find the primitive function. In other words, integration consists in finding a function whose derivative is given.

1.5.2 Objectives:

In the first two lessons, we have studied about differentiation of simple functions, logarithmic and exponential functions. In the present lesson, we shall study the

'Reverse of Differentiation' (generally called the integration or more specifically indefinite integration).

1.5.3 Definition:

If F(x) represents the differential co-efficient of f(x), i.e., $\frac{d}{dx}[f(x)] = F(x)$ then the problem of indefinite integration is to find f(x) when F(x) is given, f(x) is called the indefinite integral of F(x) with respect to x.

i.e. $\int F(x)dx = f(x)$

The function is to be integrated called integrand and is inserted between the symbols $\int$ and dx where $\int$ stand for integration and dx indicates the variables x with respect to which the integration is to be performed. The symbol $\int$ which is the integral sign is elongated S, the first letter of the word 'Sum'.

$\int F(x)dx$ is read as integral of F(x), w.r.t. x (w.r.t. is the standard symbol for 'with respect to').

1.5.4 Constant of Integration:

Whenever it is given that $\frac{d}{dx}[f(x)] = F(x)$, then integration is always written as:

$\int F(x)\, dx = \int f(x) + C$

where C being an arbitrary constant. The constant c is called the constant of integration. Let us take an example:

Suppose $f(x) = x^4$

Now $\frac{d}{dx}(x^4) = 4x^3 \quad \therefore \int 4x^3 dx = x^4$

Suppose if $f(x) = x^4 + C$

Now $\frac{d}{dx}(x^4 + C) = 4x^3 \quad \therefore \int 4x^3 dx = x^4 + C$

Since the value of C is not definitely known, the integral $\int F(x)dx$ is called an indefinite integral. The constant of integration may be added for generality.

Note: While dealing with indefinite integrals, the word indefinite is generally omitted.

1.5.5 General Rule for Integration:

In the integrand add one to the power of the variable and divide by increased power and then add a constant.

i.e. $$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C$$

$$\frac{d}{dx}\left[\frac{x^{n+1}}{n+1}\right] = \frac{(n+1)x^n}{x+1} = x^n$$

Examples:

n

(i) $$\int 3x^2 dx = 3.\frac{x^{2+1}}{3} + C = x^3 + C$$

(ii) $$\int 8x^7 dx = 8.\frac{x^{7+1}}{8} + C = x^8 + C$$

Note: The above mentioned general rule holds for values of n except for n = –1.

1.5.6 Some Standard Integrals:

The following is the list of the integrals of the standard forms:

Power Function
$$\begin{cases} \text{Rule I:} & \int x^n dx = \frac{x^{n+1}}{n+1} + C, \quad \text{provided } n \neq -1 \\ \text{Rule II:} & \int \frac{1}{x} dx = \log x + C \end{cases}$$

Exponential Function
$$\begin{cases} \text{Rule III:} & \int e^x dx = e^x + C \\ \text{Rule IV:} & \int a^x dx = \frac{a^x}{\log a} + C \end{cases}$$

Important Extensions of Standard Forms:

Extention
of Power Function

$$\begin{cases} \text{Rule V:} \int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{n+1} \times \frac{1}{a} + C, \text{ provided } n \neq -1 \\ \text{Rule VI:} \int \frac{1}{(ax+b)} dx = \log(ax+b) \times \frac{1}{a} + C \end{cases}$$

Extention of
Exponential Function

$$\left[\begin{array}{l} \text{Rule VII: } \int e^{mx+n}dx = e^{mx+n} \times \frac{1}{m} + C \\ \text{Rule VIII: } \int a^{mx+n}dx = \frac{a^{mx+n}}{\log a} \times \frac{1}{m} + C \end{array}\right.$$

Some More Results:

(i) $\int 0 \; dx = C$

(ii) $\int 1 \; dx = x + C$

(iii) $\int e^{mx}dx = \frac{e^{mx}}{m} + C$

(iv) $\int a^{mx}dx = \frac{a^{mx}}{m\log a} + C$

(v) $\int a \, dx = ax + C$

Exercise 1:

Integrate the following w.r.t. x:

(a) $\int a^{mx}dx = \frac{a^{mx}}{m \log a} + C$

(b) $\frac{}{\sqrt{x}}$

(c) $\frac{1}{2x}$

(d) $x^{3/2}$

1.5.7 Some Theorems on Integration:

1.5.7.1 Theorem 1:

If K is a constant and u is a function of x, then

$$\int kudx = k\int udx$$

i.e. The integration of the product of a constant and a function of x is equal to the product of the constant and the integral of the function.

1.5.7.2 Theorem 2:

If u and v are functions of x, then

$$\int (u+v)dx = \int udx + \int vdx$$

i.e. The integral of the sum of two functions is equal to the sum of the integral of the individual function.

1.5.7.3 Theorem 3:

$$\int (au + bv - cw + \ldots\ldots)dx = a\int udx + b\int vdx - c\int wdx + \ldots\ldots\ldots$$

i.e. The integral of the sum or difference of three or more functions is equal to the sum or difference of integrals of the individual functions.

Example 1:

Evaluate the following integrals:

$$\int (au + bv + cw \ldots)dx = a\int udx + b\int vdx + c\int wdx \ldots\ldots$$

(i) $\int (\quad x \quad)$ (ii) $\int \ldots dx$

(iii) $\int \left(\frac{1}{2\sqrt{x}} - 3^x + 4e^{2x} \right) dx$ (iv) $\int \frac{1}{\sqrt{9+4x}} dx$

Solution:

(i) $\int \left(3x^2 - \frac{2}{x} + 1 \right) dx$

$$= 3\int x^2 dx - 2\int \frac{1}{x} dx + \int 1.dx$$

$= 3\frac{x^3}{3} - 2\log|x| + x + C$, where C is an arbitrary constant.

$= x^3 - 2 \log |x| + x + C$

(ii) $\int 3x^2 dx$

$$\int 3x^2 dx = 3\frac{x^3}{3} + C$$

$= x^3 + C$

(iii) $\int \left(\frac{1}{2}\frac{1}{\sqrt[2]{x}} - 3^x + 4e^{2x} \right) dx$

$$= \frac{1}{2}\int x^{-\frac{1}{2}}dx - \int 3^x dx + 4\int e^{2x}dx$$

$$= \frac{1}{2}.\frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} - \frac{3^x}{\log 3} + 4.\frac{e^{2x}}{2} + C$$

$$= \sqrt{x} - \frac{3^x}{\log 3} + 2e^{2x} + C$$

(iv) $\int \frac{1}{\sqrt{9+4x}}dx$

$$= \int (9+4x)^{-\frac{1}{2}}dx$$

$$= \frac{(9+4x)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} \times \frac{1}{4} + C$$

$$= \frac{1}{2}.\sqrt{9+4x} + C$$

Example 2:

Evaluate the following integrals:

(i) $\int \sqrt{x}\left(x^2+3x+4\right)dx$ (ii) $\int \frac{(2x-3)^2}{x^{1/3}}dx$ (iii) $\int \frac{x}{x+1}dx$

(iv) $\int \left[\left(e^{2x}\right)^2 + \left(a^{2x}\right)^2\right]dx$ (v) $\int \frac{x^2+1}{x+1}dx$

Solution:

(i) $\int \sqrt{x}\left(x^2+3x+4\right)dx$

$$= \int x^{1/2}x^2dx + 3\int x^{1/2}xdx + 4\int x^{1/2}dx$$

$$= \int x^{5/2}dx + 3\int x^{3/2}dx + 4\int x^{1/2}dx$$

$$= \frac{x^{\frac{5}{2}+1}}{\frac{5}{2}+1} + 3.\frac{x^{\frac{3}{2}+1}}{\frac{3}{2}+1} + 4.\frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + C$$

$$= \frac{2}{7}x^{7/2} + \frac{6}{5}x^{\frac{5}{2}} + \frac{8}{3}x^{\frac{3}{2}} + C$$

(ii) $\int \frac{(2x-3)^2}{x^{1/3}} dx$

$$= \int \frac{4x^2 - 12x + 9}{x^{1/3}} dx$$

$$= 4\int x^{\frac{5}{3}} dx - 12\int x^{\frac{2}{3}} dx + 9\int x^{-\frac{1}{3}} dx$$

$$= 4.\frac{x^{\frac{5}{3}+1}}{\frac{5}{3}+1} - 12.\frac{x^{\frac{2}{3}+1}}{\frac{2}{3}+1} + 9.\frac{x^{-\frac{1}{3}+1}}{-\frac{1}{3}+1} + C$$

$$= \frac{3}{2}x^{\frac{8}{3}} - \frac{36}{5}x^{\frac{5}{3}} + \frac{27}{2}x^{\frac{2}{3}} + C$$

(iii) $\int \frac{x}{x+1} dx$

$$= \int \frac{(x+1)-1}{x+1} dx$$

$$= \int \left(1 - \frac{1}{x+1}\right) dx$$

$$= \int 1.dx - \int \frac{1}{x+1} dx$$

$$= x - \log |x + 1| + C$$

(iv) $\int \left[\left(e^{2x}\right)^2 + \left(a^{2x}\right)^2\right] dx$

$$= \int \left(e^{4x} + a^{4x}\right) dx$$

$$= \int e^{4x} dx + \int a^{4x} dx$$

$$= \frac{e^{4x}}{4} + \frac{a^{4x}}{4 \log a} + C$$

(v) $\int \frac{x^2 + 1}{x + 1} dx$

$$= \int \left(x - 1 + \frac{2}{x + 1} \right) dx$$

$$= \int x \, dx - \int 1 \, dx + 2 \int \frac{1}{x + 1} dx$$

$$= \frac{x^2}{2} - x + 2 \log|x + 1| + C$$

Exercise 1:

1. Using the definition of Indefinite Integration as the reverse of differentiation,

show that:

(i) $\int 3^{2x} dx = \frac{1}{2 \log 3} . 3^{2x} + C$

(ii) $\int e^{3x} dx = \frac{1}{3} e^{3x} + C$

2. Evaluate:

(i) $\int \left(x^7 + 7^x + 7^7\right) dx$

(ii) $\int \frac{(x + 2)^3}{x^6} dx$

(iii) $\int \frac{x + 2}{x + 1} dx$

(iv) $\int\left[\frac{2x+3}{3x+8}\right]dx$

1.5.8 Integration by Division:

If the degree of numerator is equal to or greater than the degree of denominator, divide the numerator by the denominator till the degree of the remainder is less than of the divisor.

Example:

(i) $\int\frac{x+3}{x+5}dx$

$=\int\left(1-\frac{2}{x+5}\right)dx \qquad =\int 1dx-\int\frac{2}{x+5}dx$

= x – 2 log (x + 5) + C

(ii) $\int\frac{x^3}{x+1}dx$

$=\int\left(x^2-x+1-\frac{1}{x+1}\right)dx$

$=\frac{x^3}{3}-\frac{x^2}{2}+x-\log(x+1)$

To integrate an expression of form $\frac{1}{\sqrt{ax+b}\pm\sqrt{ax+c}}$

First rationalise the denominator and then integrate.

Example 2:

Evaluate $\int\frac{1}{\sqrt{x+4}+\sqrt{x+3}}dx$

Solution:

We have $\int\frac{1}{\sqrt{x+4}+\sqrt{x+3}}$

By rationalisation

$=\frac{1}{\sqrt{x+4}+\sqrt{x+3}}\cdot\frac{\sqrt{x+4}-\sqrt{x+3}}{\sqrt{x+4}-\sqrt{x+3}}$

$$= \frac{\sqrt{x+4}-\sqrt{x+3}}{\left(\sqrt{x+4}\right)^2-\left(\sqrt{x+3}\right)^2}$$

$$= \frac{\sqrt{x+4}-\sqrt{x+3}}{x+4-x-3} = \sqrt{x+4}-\sqrt{x+3}$$

$$\therefore \quad \int \frac{1}{\sqrt{x+4}+\sqrt{x+3}}dx = \int \sqrt{x+4}-\sqrt{x+3}\,dx$$

$$= \int (x+4)^{1/2}dx - \int (x+3)^{1/2}dx$$

$$= \frac{(x+4)^{3/2}}{3/2} - \frac{(x+3)^{3/2}}{3/2}$$

$$= \frac{2}{3}\left[(x+4)^{3/2}-(x+3)^{3/2}\right]$$

Exercise 2:

Integrate

(i) $\int e^{4x}$

(ii) $\int \frac{x^2-1}{x+1}dx$

(iii) $\int \sqrt{x}e^x dx$

(iv) $\frac{x^2-x-6}{x+2}dx$

(v) $\frac{dx}{\sqrt{x+1}-\sqrt{x}}$

1.5.9 Methods of Integration:

When the function to be integrated is not in the standard form, we have to adopt the following methods of integration:

1. Integration by Substitution
2. Integration by Parts
3. Integration by Partial Fractions

1.5.9.1 Integration by Substitution:

If the integrand is not of the simple (or standard) form, we may use substitution

to change the variable and ultimately to reduce the integral into the standard form.

Example:

Prove that

(i) $\int f(ax+b)dx = \frac{1}{a}\int f(z)\,dx$

(ii) $\int \{f(x)\}^n . f'(x)dx = \frac{\{f(x)^{n+1}\}}{n+1} + C,$ provided $n \neq -1$

(iii) $\int \frac{f'(x)}{f(x)}dx = \log|f(x)| + C$

Solution :

(i) $\int f(ax+b)dx$

Put ax + b = z

Take derivative

$$a.1 = \frac{dz}{dx}$$

$$\therefore \quad dx = \frac{dz}{a}$$

$$\therefore \quad \int f(ax+b)dx = \int f(z)\frac{1}{a}dz$$

$$= \frac{1}{a}\int f(z)dz$$

(ii) $\int \{f(x)\}^n . f'(x)dx$

Put f(x) = z

then $f'(x) = \frac{dz}{dx}$

or f ' (x) dx = dz

$$\therefore \quad \int \{f(x)\}^n . f'(x)dx = \int z^n . dz = \frac{z^{n+1}}{n+1} + C$$

$$= \frac{\{f(x)\}^{n+1}}{n+1} + C$$

(iii) $\int \frac{f'(x)}{f(x)} dx$

Put $f(x) = z$

$\therefore$ $f'(x)dx = dz$

or $dx = \frac{dz}{f'(x)}$

$\therefore$ $\int \frac{f'(x)}{f(x)} dx = \int \frac{dz}{z} = \log|z| + C$

$= \log [f(x)] + C$

Example 2:

Integrate the following:

(i) $\int \frac{(\log x)^3}{x} dx$

(ii) $\int x\sqrt{1+x^2}\, dx$

Solution:

(i) $\int \frac{(\log x)^3}{x} dx$

Let $\log x = t$, so that $\frac{1}{x} dx = dt$

$\therefore$ $\int \frac{(\log x)^3}{x} dx = \int t^3 dt$

$$= \frac{t^4}{4} + C = \frac{(\log x)^4}{4} + C$$

(ii) $\int x\sqrt{1+x^2}\, dx$

Put $1 + x^2 = t$

$\therefore$ $2x\, dx = dt$

or $\quad xdx = \frac{dt}{2}$

$\therefore \quad \int x\sqrt{1+x^2}dx = \int \sqrt{t}\frac{dt}{2} = \frac{1}{2}\int t^{1/2}dt$

$$= \frac{1}{2}.\frac{t^{3/2}}{3/2} + C$$

$$= \frac{2}{3} \times \frac{1}{2}t^{3/2} + C = \frac{1}{3}\left(1+x^2\right)^{3/2} + C$$

Example 3:

(i) $\int \frac{(\log x)^2}{x}dx$ (ii) $\int \frac{e^x}{e^x+1}dx$

Solution:

(i) $\int \frac{(\log x)^2}{x}dx$

$$= \int \frac{1}{x}.(\log x)^2 dx$$

$$= \frac{(\log x)^{2+1}}{2+1} \qquad \left\{\text{By applying } \int [f(x)]^n . f'(x)\right\}$$

$$= \frac{1}{3}(\log x)^3$$

(ii) $\int \frac{e^x}{e^x+1}dx$

$= \log\ (1 + e^x)$ $\{\because$ num is the diff. co – eff. of denom.$\}$

$$\left\{\text{By applying } \int \frac{f'(x)}{f(x)}dx\right\}$$

Exercise 3:

Evaluate the following integrals:

(i) $\int \frac{e^x + e^{-x}}{e^x e^x} dx$ (ii) $\int \frac{x}{1-x^2} dx$

More Formulae:

(i) $\int e^{f(x)}.f'(x)dx = e^{f(x)}$

(ii) $\int a^{f(x)}.f'(x)dx = -\frac{a^{f(x)}}{\log a}$

Example 1:

Evaluate the following:

(i) $\int \frac{e^{\sqrt{x}}}{\sqrt{x}} dx$ (ii) $\int a^{x^3}.x^2 dx$

Solution:

(i) $\int \frac{e^{\sqrt{x}}}{\sqrt{x}} dx$

$= 2\int \frac{e^{\sqrt{x}}}{2\sqrt{x}} dx \quad \left\{\because \frac{1}{2\sqrt{x}} \text{ is the diff. co-eff. of } \sqrt{x}\right\}$

$= 2e^{\sqrt{x}} \quad \left\{\text{using} \int e^{f(x)}.f'(x)dx\right\}$

(ii) $\int a^{x^3}.x^2 dx$

Now $\int a^{x^3}.x^2 dx = \frac{1}{3}\int (3x^2) a^{x^3} dx \quad \left\{\because 3x^2 \text{ is the diff. co-eff. of } x^3\right\}$

$= \frac{1}{3}\frac{a^{x^3}}{\log a} \quad \left\{\text{using} \int a^{f(x)}.f'(x)dx\right\}$

1.5.9.2 Integration by Parts (Integration of the Product of Two Functions):

Rule: If u and v be two functions of x such that u is differenitable and v is integrable, then

$$\int (u.v)dx = u\int v dx - \int \left\{\frac{du}{dx}.\int v dx\right\} dx$$

Integral of the product of two functions = (1st Function) × (Integral of 2nd Function) minus

[Integral of (Derivative of 1st Function × Integral of 2nd Function)]

Solution:

If U and V be two functions of x, then

$$\int (u.v)dx = u\int vdx - \int\left\{\frac{du}{dx}.\int vdx\right\}dx$$

Integrating both sides, we get

$$uv = \int u.\frac{dv}{dx}dx + \int v.\frac{du}{dx}dx$$

i.e. $\int u.\frac{dv}{dx}dx = uv - \int v.\frac{du}{dx}dx$

If $u = f(x), \frac{dv}{dx} = g'(x)$

Then $\int \frac{dv}{dx}dx = \int g'(x)dx$ i.e., $v = \int g'(x)dx$

$\therefore$ (i) can be written as

$$\int f(x)g'(x)dx = fx\int g'(x)dx - \int\left[\int g'(x)dx\frac{d}{dx}fx\right]dx$$

Note: While solving problems of Integration by Parts, following rules may be followed:

1. Choose the first and second functions in such a way that the differential co-efficient of the first function and integral of the second function can be easily found.
2. If one function of the product is a power function and the second function is a logarithmic function, take always logarithmic function as first one and power function as second one.
3. If the integrand consists of a single logarithmic function, then take always logarithmic function as the first function and 1 as the second function.
4. Rule of n integration by parts may be used repeatedly if required.

Example 1: Evaluate $\int\left\{\frac{d}{dx}(x).\int 2^x dx\right\}dx$

(i) $\int x.2^x dx$ (ii) $\int \frac{\log x}{x^2}dx$

(iii) $\int x^n \log x dx$ (iv) $\int x^3 e^{x^2} dx$

Solution:

(i) $\int x.2^x dx$

$$= x\int 2^x dx - \left\{\frac{d}{dx}(x).\int 2^x dx\right\}dx$$

$$= x\frac{2^x}{\log 2} - \int\left\{1.\frac{2^x}{\log 2}\right\}dx + C$$

(ii) $\int \frac{\log x}{x^2} dx$

$$= \log x\int \frac{1}{x^2} dx - \int\left\{\frac{d}{dx}(\log x).\int \frac{1}{x^2} dx\right\}dx$$

$$= \log x.\left[-\frac{1}{x}\right] - \int\left[\frac{1}{x}\times -\frac{1}{x}\right]dx + C$$

$$= -\frac{\log x}{x} + \int \frac{1}{x^2} dx + C = -\frac{\log x}{x} - \frac{1}{x} + C$$

$$= -\frac{1}{x}(\log x + 1) + C$$

(iii) $\int x^n \log x dx$

$$= \int \log x.x^n dx$$

$$= \log x\int x^x dx - \int\left\{\frac{d}{dx}(\log x)\int x^x dx\right\}dx$$

$$= \log x.\frac{x^{n+1}}{n+1} - \int\left[\frac{1}{x}\times\frac{x^{n+1}}{n+1}\right]dx + C$$

$$\frac{x^{n+1}}{n+1}.\log x - \frac{1}{n+1}\int x^n dx + C$$

$$= \frac{x^{n+1}}{n+1}.\log x - \frac{1}{n+1}\frac{x^{n+1}}{n+1} + C$$

$$= \frac{x^{n+1}}{n+1}\left[\log x - \frac{1}{n+1}\right] + C$$

(iv) $\int x^3 e^{x^2} dx$

Put $x^2 = z$, $2x\ dx = dz$

$$= \int x^3.e^{x^2} dx = \int \frac{x^2}{2}.2xe^{x^2} dx = \int \frac{x^2}{2} e^{x2}(2x)dx$$

$$= \int \frac{z}{2} e^2 dz = \frac{1}{2}\int ze^z dz$$

1.5.9.3. Integration by Partial Fractions:

When the degree of numerator f(x) is less than the degree of denominator F(x), then the given function $\frac{f(x)}{F(x)}$ is first split into partial fractions which are reduced to standard forms and then integrate.

Example 1:

Evaluate

$$\int \frac{2x+1}{(x+1)(x-1)} dx$$

Solution:

Let $I = \int \frac{2x+1}{(x+1)(x-1)} dx$

Let $\frac{2x+1}{(x+1)(x-1)} = \frac{A}{x+1} + \frac{B}{x-1}$ .....(i)

or $(2x + 1) = A (x - 1) + B (x + 1)$ .....(ii)

To find A, put $x + 1 = 0$, i.e. $x = -1$, in (ii)

$2 (-1) + 1 = A (-1 -1) + B (-1+1)$

$-1=-2A \Rightarrow A = \frac{1}{2}$

To find B, put x – 1 = 0, i.e. x = +1, in (ii) we get 3=B.2

$$\Rightarrow B = \frac{3}{2}$$

$$\therefore \quad \frac{2x+1}{(x+1)(x-1)} = \frac{1}{2(x+1)} + \frac{3}{2(x-1)}$$

$$\therefore \quad I = \int \frac{2x+1}{(x+1)(x-1)}\,dx = \int \frac{1}{2(x+1)}\,dx + \int \frac{3}{2(x-1)}\,dx$$

$$= \frac{1}{2}.\int \frac{1}{x+1}dx + \frac{3}{2}\int \frac{1}{x-1}dx$$

$$= \frac{1}{2}\log(x+1) + \frac{3}{2}\log(x-1)$$

Exercise 4:

1. Evaluate the following:

(i) $\int \frac{(x-1)}{(x-2)(x-3)}\,dx$ (ii) $\int \frac{xe^{2x}}{(2x+1)^2}\,dx$ (iii) $\int \frac{dx}{4x^2-9}$

(iv) $\int \frac{x^3}{(x^2+3)^3}\,dx$ (v) $\int \frac{dx}{x(x^4+1)}$

2. $\int (2x+5)\sqrt{x^2+5x}\,dx$

3. $\int e^x\left(\frac{1}{x} - \frac{1}{x^2}dx\right)$

4. $\int \left(\frac{1}{\log x} - \frac{1}{(\log x)^2}\right)dx$

Some More Examples:

Example 1:

Evaluate

$$\int \frac{dx}{\left(1+e^{x}\right)\left(1-e^{-x}\right)}$$

Solution:

$$\int \frac{dx}{\left(1+e^{x}\right)\left(1-e^{-x}\right)} = \int \frac{dx}{e^{x}-e^{-x}} = \int \frac{e^{x}dx}{e^{2x}-1}$$

Put $e^x = t$, i.e. $e^x\,dx = dt$

$$\therefore \quad \int \frac{e^{x}dx}{e^{2x}-1} = \int \frac{dt}{t^{2}-1}$$

$$= \frac{1}{2}\log\left(\frac{t-1}{t+1}\right) = \frac{1}{2}\log\left(\frac{e^{x}-1}{e^{x}+1}\right) + C$$

Example 2:

Determine $\int \frac{6-x}{(x-3)(2x+5)}dx$ by the method of Partial Fractions.

Solution: 1

Let $\frac{6-x}{(x-3)(2x+5)} = \frac{A}{x-3} + \frac{B}{2x+5}$ 11

$$\Rightarrow 6-x = A(2x+5) + B(x-3)$$

Comparing co-efficients of equal powers of x on both sides, we get

$2A + B = -1$

$5A - 3B = 6$

i.e. $A = \frac{3}{11}, B = -\frac{17}{11}$

$$\therefore \quad \int \frac{(6-x)\,dx}{(x-3)(2x+5)} = \int \frac{3}{11(x-3)}dx - \int \frac{17}{11(2x+5)}dx$$

$$= \frac{3}{11}\log(x-3) - \frac{7}{22}\log(2x+5) + C$$

1.5.10 Summary :

In the present lesson, the concept of integration (indefinite integration), constant of integration, general rules of integration, some standard theorems on integration have been explained with illustrations. Three important methos of integration have also been given alongwith illustrations.

1.5.11 Recommended Books :

1.	S.C. Aggarwal and R.K. Rana	:	Basic Mathematics for Economists
2.	D. Bose	:	Introduction to Mathematical Economics
3.	B.M. Aggarwal	:	Mathematics for Business and Economics
4.	Aggarwal and Joshi:		Mathematics for Students of Economics

1.6.12 List of Questions:

1.6.12.1 Short Questions:

1. Evaluate the following integrals:

(i) $\int 5x^7 dx$ (ii) $\int x^3 + 3x^2 - 4x + 5$

(iii) $\int x\sqrt{x^4}\, dx$ (iv) $\int \frac{1}{x^7} dx$

2. Give Meaning/Definition of Integral.
3. State Three Methods of Integratio.

1.5.12.2 Long Questions:

Evaluate:

1. $\int x - 1\sqrt{x+2}\, dx$
2. $\int e^{3x} + \frac{\log x}{x}$
3. $\int \log x^x$
4. $\int \log(5+x)\, dx$
5. $\int \frac{x+3}{1-x^2} dx$

6. $\int \frac{dx}{2x^2 + x - 1}$

7. $\int \frac{1}{x^2 + 5x + 6} dx$

8. $\int \frac{2x + 1}{(x+1)(x-1)} dx$

9. $\int (\log x^2) dx$

10. $\int \frac{1+x}{(2+x)^2} e^x \, dx$

- This lesson has been written from the assistance taken from Distance Education Council.

LESSON NO. 1.6 **AUTHOR : DR. VIPLA CHOPRA**

ANALYSIS OF CONSUMER'S SURPLUS AND PRODUCER'S SURPLUS

Structure:

1.6.1 Introduction:

In previous lesson we have studied the method of finding out integration more specifically indefinite integration. It is the reverse of differentiation which deals with the situation where the derivative is given and we have to find the primitive function. After dealing with indefinite (general) integrals, we now come to another type of integrals which have unique value and are called definite integrals.

The process of integration has very useful application in various fields of economics like in the analysis of Consumer's Surplus Producer's Surplus etc.

1.6.2 Objectives:

After reading this lesson you should be able to:

* define definite integral
* explain the concept of consumer's surplus
* distinguish between consumer's surplus under, state competition and under monopoly

* acquaint yourself with the concept of producer's surplus
* know some more application of integration in economics

1.6.3 Definite Integral

The definite integral of a function f(x) in the closed interval (a, b) denoted by $\int_a^b f(x)dx$ is equal to the difference in the value of the integral of f(x)dx for two assigned values say a and b of the independent variable x and is written as

$$\int_a^b f(x)dx = \left[F(x)+c\right]_a^b = F(b)-F(a)$$

where F(x) + c = $\int_a^b f(x)dx$

a is called the lower limit and b is called the upper limit of integration (a < b). The interval (a, b) is called the range of integration.

1.6.3.1 Rule to Evaluate Definite Integral

To evaluate a definite integral such $\int_a^b f(x)dx$ as process is as under:

(i) Find the indefinite integral $\int f(x)dx = F(x)+c$ say.

(ii) Put x = b (upper limit) and get F(b) + c

(iii) Put x = a (lower limit) and get F(a) + c

(iv) Subtract (iii) from (ii) and get F(b) – F(a).

Note: The constant of integration does not play any role in definite integral.

Example 1:

Evaluate the following integrals:

(i) $\int_1^2 1+\frac{2}{\sqrt{x}}+3xdx$ (ii) $\int_1^2 \frac{\left(t^2+2t+5\right)}{t}dt$

Solution:

(i) $$\int_1^2\left[1+\frac{2}{\sqrt{x}}+3x\right]dx = \left[x+\frac{2x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}+\frac{3x^2}{2}\right]_1^2$$

$$= \left[x + 4x^{\frac{1}{2}} + \frac{3}{2}x^2 \right]_1^2$$

$$= \left[2 + 4\sqrt{2} + \frac{3}{2} \times 4 \right] - \left[1 + 4 + \frac{3}{2} \right] = \left(8 + 4\sqrt{2} \right) - \frac{13}{2}$$

$$= \frac{3}{2} + 4\sqrt{2}$$

(ii) $$\int_1^2 \frac{\left(t^2 + 2t + 5\right)}{t} dt = \int_1^2 \left[t + 2 + \frac{5}{t} \right] dt = \left[\frac{t^2}{2} + 2t + 5\log t \right]_1^2$$

$$= \frac{2^2}{2} + 2.2 + 5\log 2 - \left[\frac{1}{2} + 2 + 5\log 1 \right]$$

$$= 2 + 4 + 5\log 2 - \left[\frac{5}{2} + 5 \times 0 \right] = 6 - \frac{5}{2} + 5\log 2 = \frac{7}{2} + 5\log 2$$

Example 2:

Evaluate :

(i) $\int_0^1 \frac{(x+1)(x-4)}{\sqrt{x}}$ (ii) $\int_4^6 \frac{dx}{x^2 - 9}$ (iii) $\int_0^1 \frac{\log(1+x)}{1+x} dx$

Solution:

(i) $$\int_0^1 \frac{(x+1)(x-4)}{\sqrt{x}} = \int \left[\frac{x^2 - 3x - 4}{\sqrt{x}} \right] dx$$

$$= \int \left(x^{3/2} - 3x^{1/2} - 4x^{-1/2} \right) dx = \frac{2}{5}x^{5/2} - 2x^{3/2} - 8x^{1/2} + C$$

(ii) $$\int_4^6 \frac{dx}{x^2 - 9} = \int_4^6 \frac{dx}{x^2 - 3^2}$$

$$= \frac{1}{2.3} \left[\log \left| \frac{x-3}{x+3} \right| \right]_4^6 = \frac{1}{6} \left[\log \frac{3}{9} - \log \frac{1}{7} \right]$$

$$= \frac{1}{6}\log\left[\frac{1}{3}\times\frac{7}{1}\right] = \frac{1}{6}\log_c\frac{7}{3}$$

(iii) $$\int_0^1 \frac{\log(1+x)}{1+x}dx = \left[\log(1+x).\int\frac{dx}{1+x} - \int\left\{\frac{d}{dx}\left[\log(1+x)\right].\int\frac{1}{1+x}dx\right\}dx\right]_0^1$$

$$\left[\int\frac{\log(1+x)}{1+x}dx\right\}dx\right]_0^1$$

$$= \left[\log(1+x).\log(1+x)\right]_0^1 - \int_0^1 \frac{1}{1+x}\log(1+x)dx$$

or $$2\int_0^1 \frac{\log(1+x)}{1+x}dx = (\log 2)^2 - 0 \qquad (\because \log 1 = 0)$$

or $$\int_0^1 \frac{\log(1+x)}{1+x}dx = \frac{1}{2}(\log 2)^2$$

Example 3:

Show that $\int_1^4 \sqrt{x}\,dx = \frac{14}{3}$

Solution:

$$\int_1^4 \sqrt{x}\,dx = \int_1^4 x^{1/2}dx = \left[\frac{x^{3/2}}{\frac{3}{2}}\right]_1^4$$

$$= \frac{2}{3}\left[x^{3/2}\right]_1^4 = \frac{2}{3}.\left[4^{3/2} - 1^{3/2}\right]$$

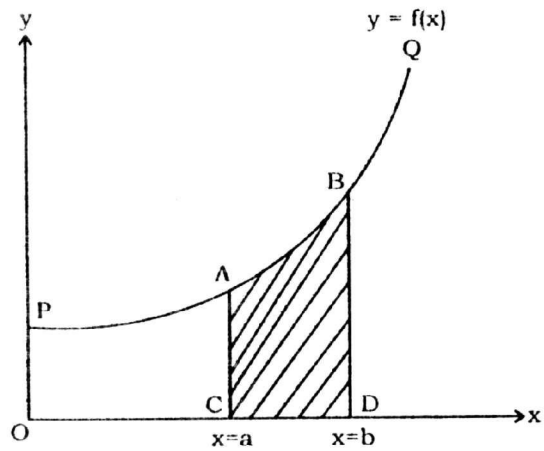
$$= \frac{2}{3}\left[\left(2^2\right)^{3/2} - 1\right] = \frac{2}{3}[8-1] = \frac{14}{3}$$

1.6.3.2 Definite Integral as an Area under a Curve

Every definite integral has a definite value. That value may be interpreted geometrically to be a particular area under a given curve.

Let y = f (x) be an increasing function of x in a < x < b and let if PQ be the continuous curve for y = f (x). Let AC and BD be the ordinates at the points

x = a and x = b.



The area under the curve y = f(x) between the limits a and b can written as a definite integral.

Area under the curve = $\int_a^b f(x)dx$

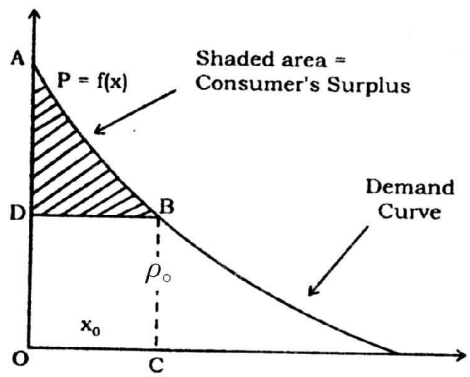
Now we will make use of definite integral to find Consumer's surplus and producers surplus.

1.6.4 Consumer's Surplus (CS)

First we shall study the Concept of Consumer's Surplus.

Definition : In the words of Dr. Marshall 'The excess of price which a person would be willing to pay rather than go without the thing over that which he actually does pay is the economic measure of this surplus of satisfaction. It may be called Consumer's Surplus. It is estimated by the area between the demand curve and the price which is actually paid.

Let P = f(x) be a demand function where y is the price per unit and x, the number of units. Let a buyer purchases x_0 units at a price p_0 then the total amount paid by the consumer = x_0 p_0. But the total satisfaction derived by the consumer = Area under the demand curve between x = 0 and x = x_0. In Fig. 1 area OABC represents the area under the demand curve between x = 0 and x = x_0.



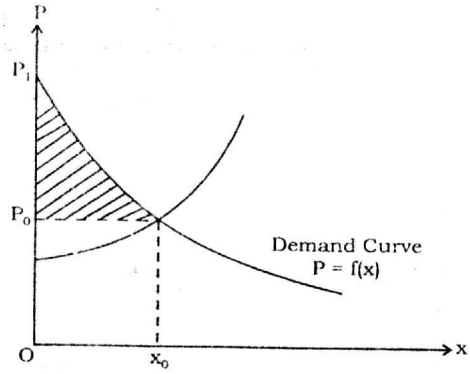
The area under the demand curve from x = 0 to x = x_0 = $\int_{x}^{x_0} f(x)dx$

$\therefore$ Consumer's surplus = $\int_{x}^{x_0} f(x)dx - x_0P_0$

CS= what a consumer is prepared to pay what he actually pays.

Alternative Method

There is an another method to find consumer's surplus by integrating the demand function (as a function of price) with respect to Price.



Consumer's Surplus = $\int_{P_0}^{P_1} (\text{demand function as a function of price})\ dP,$

where P_1 is the price at which quantity demanded is zero. It can be obtained by putting x = 0 in the demand function.

1.6.4.1 Consumer's Surplus under Pure Competition

Under Pure Competition the point of intersection of demand and supply curves gives the equilibrium price and quantity purchased.

Step 1: Obtain x_0 i.e. quantity purchased by solving the demand and supply curve equations.

Step 2: Substitute this value of x_0 in the demand equation the corresponding value of P_0 is obtained.

1.6.4.2 Consumer's Surplus Under Monopoly

Under Monopoly Consumer's Surplus is obtained as follows :-

Step 1: Determine Quantity purchased x_0 by equating MR and MC.

Step 2: Substitute the value of x_0 in demand equation. Obtain price

Step 3: Obtain Consumer's surplus

$$CS = \int_0^{x_0} f(x)dx - P_0x_0$$ where P = f(x) is the demand function.

Example 1:

If the market demand curve is p = 20 – 2x, where p and x are respectively the price and demand of commodity, find the consumer's surplus when p = 4 and p = 8.

Solution : Given demand function is p = 20 – 2x.

When p = 4, 4 = 20 – 2x or x = 8.

When p = 8, 8 = 20 – 2x or x = 6.

Consumer's Surplus = $\int_0^x f(x)dx - Px$

When p = 4, Consumer's Surplus = $\int_0^8 (20-2x)\,dx - 4\times 8$

$$= \left(20x - x^2\right)_0^8 - 32$$

$$= 20 \times 8 - 8^2 - 32 = 64$$

When p = 8, Consumer's Surplus = $\int_0^x f(x)dx - Px$

$$= \int_0^6 (20-2x)dx - 8\times 6 = \left(20x - x^2\right)_0^6 - 48$$

$= 20 \times 6 - 6^2 - 48 = 36$

Example 2 : Find the customer's surplus for the demand function p = 25 – 2x where x_0 = 10.

Solution : Customer's surplus = $\int_0^{x_0} P dx - P_0 x_0$

Given P = 25 – 2x
when x = 10, P = 5

$$\int_0^x (25-2x)dx - 5\times 10 = \left[25x - x^2\right]_0^{10} - 50 = \left[25\times 10 - 100\right] - [25\times 0 - 0] - 50 = 100$$

Example 3:

A monopolist demand function is x = 210–3 P where x is the quantity demanded when price is Rs. p per unit with the average cost function AC (x) = $x + 6 + \frac{10}{x}$

Find the Consumer's Surplus at the price which monopolist would like to fix.

Solution : Under monopoly

MR = MC

$x = 210 - 3P \Rightarrow 3P = 210 - x \Rightarrow P = \frac{210 - x}{3}$

Revenue = xp = $\frac{x(210 - x)}{3}$

$MR = \frac{210 - 2x}{3}$

$AC = x + 6 + \frac{10}{x}$

$TC = x^2 + 6x + 10 \quad \left[\because \text{Total Cost} = x(AC)\right]$

MC = 2x + 6

Equate MR = MC $\Rightarrow \frac{210 - 2x}{3} = 2x + 6$

210 – 2x = 6x + 18

$\Rightarrow 8x = 192 \Rightarrow x = 24$

When x = 24,

24 = 210 – 3P $\Rightarrow$ P = 62

Consumer's Surplus =

$$\frac{1}{3}\int_0^{24}(210-x)dx - 24\times 62 = \frac{1}{3}\left[210x - \frac{x^2}{2}\right]_0^{24} - 24\times 62 = 96$$

Example 4:

The marginal cost function of a firm is given by MC = $3000e^{0.3x}$ + 50, when x is quantity produced. If fixed cost is Rs. 80,000, find the total cost function of the firm.

Solution : Given that

MC = $3000e^{0.3x}$ + 50 and C(0) = 80,000

Integrating the marginal cost function, we obtain

$$C(x) = \int\left(3000e^{0.3x} + 50\right)dx + k$$

= $10000e^{0.3x}$ + 50x + k

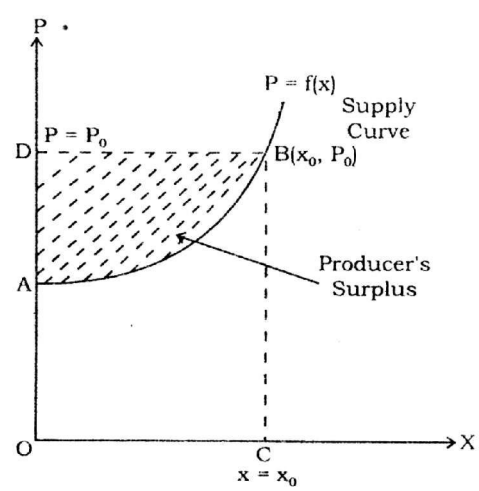
$\Rightarrow$ C(0) = 10000 + k = 80000 $\Rightarrow$ k = 70000

Thus total cost function of the firm is given by

C(x) = $10000e^{0.3x}$ + 50x + 70000

1.6.5 Producer's Surplus

Sometimes, it so happens that a producer is willing to sell his product at a certain price but when he actually sells the article he gets more than his expectation. In such cases the difference between the price at which the producer would be willing to sell his product and the price he actually receives is called Producer's Surplus.



Figure

Suppose producer sells x_0 limits at a price P_0, then the amount which expects is given by the area under the supply curve from x = 0 to x = x_0 and the amount which he actually receives = $x_0 P_0$.

In fig. OABC is the area under the supply curve from x = 0 to x = x_0

Area OABC = $\int_0^{x_0} f(x)\,dx$

Amount actually received by the producer = $x_0 P_0$
= Area ODBC

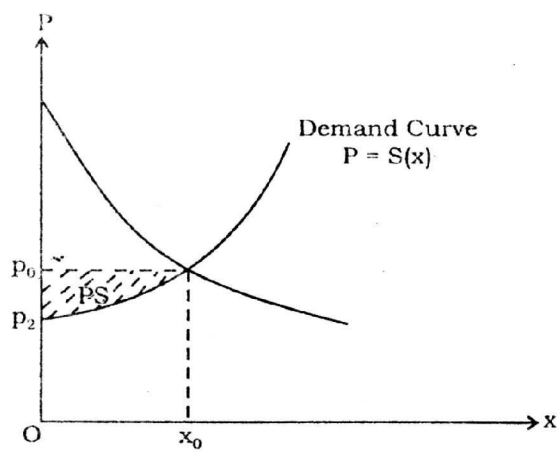
Producer's Surplus = $x_0 P_0 - \int_0^{x_0} f(x)dx$

Alternatively

Producer's Surplus = Integrate the supply function (as a function of p) with respect to p.

$$PS = \int_{P_2}^{P_0} (\text{Supply function as a function of p})\,dp.$$

where p_2 is the price obtained by setting x = 0 in the supply function.



Example 1:

The supply function of a product is $y = 3x^2 + 6$. Find the producer's

surplus when 10 units are supplied.

When x = 10, y = 3x 100 + 6 = 306

Solution:

$\therefore$ Producer's Surplus = 306 × 10 – $\int_0^{10}(3x^2+6)\,dx$

$$= 3060 - \left[\frac{3x^3}{3} + 6x\right]_0^{10}$$

= 3060 – 1000 – 60 = Rs. 2000

Example 2:

Given the demand function P = 20 – 5x and the supply function P = 4 + 3x, find the Consumer's Surplus (CS) and Producer's Surplus PS).

Solution: Here

Demand function is P = 20 – 5 x .....(i)

Supply function is P = 4 + 3x .....(ii)

Solving (i) and (ii), we get

20 – 5 x = 4 + 3x or x = 2

$\therefore x_0 = 2$

and $P_0 = 20 - 5x_0 = 20 - 5 \times 2 = 10$

$$CS = \int_0^2 (20-5x)\,dx - 20 = 10$$

and $$PS = 20 - \int_0^2 (4+3x)\,dx = 6$$

Exercise 1:

1. The supply curve for a commodity is $p = \sqrt{9+x}$ and the quantity sold is 7 units. Find the Producer's Surplus. Can you find the Consumer's Surplus ? If yes find it, if not explain with the help of diagram why not.
2. Find the Producer's Surplus for the Supply function $p^2 - x = 9$ when x = 7
3. Under pure competition for a commodity the demand and supply laws are

$$P_d = \frac{8}{x+1} - 2, \text{ and } P_S = \frac{1}{2}(x+3) \text{ respectively.}$$

Determine the Consumer's Surplus and the Producer's Surplus.

1.6.6 More Application of Integration in Economics

1. To find Total Revenue for selling x units when Marginal revenue is given

Total Revenue = $\int (\text{Marginal Revenue})dx + K$

Where K is the constant of integration and its value can be found by puttng the initial condition viz., TR = 0 when q = 0

Example: If the marginal revenue function is MR = 8 – 6p – $2q^2$, determine the revenue and demand function.

Solution : Given M = 8 – 6q – $2q^2$

$$TR = \int MRdq + K$$

$$= \int (8 - 6q - 2q^2)dq + K$$

$$\therefore TR = 8q - \frac{6q^2}{2} - \frac{2q^3}{3} + K$$

or $TR = 8q - 3q^2 - \frac{2}{3}q^3 + K$

Initial Condition, when q = 0, TR = 0

$\therefore$ K = 0

$$TR = 8q - 3q^2 - \frac{2q^3}{3}$$

2. To find the total cost when marginal cost is given

Total cost for producing x units = c (x) = $\int MC(x)\,dx + K$, where k is constant of integration and is determined from the initial conditions given in the question.

3. To find the maximum profit when Marginal Revenue and Marginal Cost Functions are given

Profit = TR – TC

where TR = $\int MR\,dq + K$ and TC = $\int MCdq + K$

i.e. Total Profit = $\int (\text{Marginal Revenue}) - (\text{Marginal Cost})\ dx + K$

To find output that maximises total profits, we equate MR = MC.

Example 1 : The marginal cost and marginal revenue of a firm are given as MC = 4 + 0.08 x and MR = 12. Compute the total profits, given that TC at output zero is 0.

Solution : Given MC = 4 + 0.08x, MR = 12

$$\therefore TC = \int MC(x)\,dx + K_1 = \int (4 + 0.8x)\,dx + K_1$$

$$TC = 4x + \frac{0.08x^2}{2} + K_1$$

$\therefore$ TC = 4x + 0.04x^2 + K_1

Initial Condition

When x = 0, TC = 0 $\therefore$ K_1 = 0

$\therefore$ TC = 4x + 0.04x^2

TR = $\int$ MR dx + K = $\int$ 12 dx + K

TR = $\int$ MR dx + K = $\int$ 12 dx + K

Initial Cond

When x = 0, TR = 0, $\Rightarrow$ K_1 = 0

$\therefore$ TR = 12x

Profits = TR – TC

P = 12x – 4x – 0.04x^2

At equilibrium, MR = MC

$\therefore$ 4 + 0.08x = 12

0.08x = 8

x = 100

At x = 100

$\therefore$ Profits = 12 × 100 – 4 × 100 – 0.04 (100)^2

= 1200 – 400 – .04 × 10000

= 800 – 400 = 400

4. To Find Consumption Function from Marginal Propensity to Consume

Consumption function is given by $C = \int MPCdy + K$

Here, C is the consumption and Y be the level of income, K is an arbitrary constant. The value of K can be determined by putting the initial condition.

Example 1 : Find the consumption function given that income equal Consumption when Y = 100, and that marginal prosperity to consume $\frac{dC}{dY} = 0.7$, where Y is income and C is consumption.

Solution : The marginal propensity to consume is

$$\frac{dC}{dY} = 0.7$$

$C = \int 0.7dy + K,$ where K is constant of integration

= 0.7Y + K (i)

Where y = 100, then Y = C $\therefore$ 100 = 0.7 × 100 + K

$\therefore$ K = 30

$\therefore$ from (i) C = 0.7Y + 30, which is the required consumption function.

5. To find demand function if elasticity of demand is given

Price elasticity of demand = e_p = $-\frac{p}{x}.\frac{dx}{dp}$

$$\Rightarrow e_p.\frac{dp}{p} = -\frac{dx}{x}$$

$$\Rightarrow e_p \int \frac{dp}{p} = -\int \frac{dx}{x} + \log K$$

$$\Rightarrow e_p \log P = -\log x + \log K$$

$$\Rightarrow \log P^{(e_p)} + \log x = \log K$$

$$\Rightarrow \log xP^{e_p} = \log K$$

$$\Rightarrow xP^{(e_p)} = K$$

Example 1 : Obtain the demand function for a commodity whose elasticity of demand is given as $e_{x_p} = a - bP,$ where a and b are constants and P denotes the price per unit of the commodity.

Solution : Elasticity of demand $= -\frac{P}{x}.\frac{dx}{dP} = a - bP$

$$\Rightarrow -\frac{dx}{x} = \left[\frac{a - bP}{P}\right] dP$$

$$= \left[\frac{a}{P} - b\right] dP$$

Integrating both sides we get

$-\log x = a \log P - bP + C$

$\Rightarrow \log x = -\log P^a + \log_e e^{-bP+C} \quad (\because \log_e e^x = x \log_e e = x)$

$= \log P^{-a} e^{-bP+C}$

$\Rightarrow x = P^{-a} e^{-bP+C}$, where C is constant of integration.

6. To find capital formation from investment function.
7. In compound interest calculations when compounding is done continuously.

1.6.7 Summary

In the present lesson we have studied the use of definite integration in finding Consumer's Surplus under:

(i) Pure competition and (ii) under monopoly. Besides this integration is useful in finding total cost (given marginal cost), total revenue (given marginal revenue), maximum profit (given marginal revenue and marginal cost, and producer's surplus (given supply function) etc. Thus the technique of integraiton has very useful application in the field of Business and Economics.

1.6.8 Recommended Books :

1.	Dr. S.C. Aggarwal and Dr. R.K. Rana	:	Basic Mathematics for Economists
2.	Dr. D. Bose	:	Introduction of Mathematical Economics
3.	Dr. B.M. Aggarwal	:	Mathematics Economics
4.	Dr. Aggarwal and Joshi	:	Mathematics for Students of Economics

1.6.9 List of Questions

1.6.9.1 Short Questions

1. Evaluate (i) $\int_0^2 (2x+7)\,dx = 18$ (ii) $\int_2^2 3X\,dx$
2. If MC = $3 - 2x - x^2$, find the Total Cost.
3. If the Marginal Revenue Function MR = 100 – 4 Q, find the Total Revenue Function.
4. Define Consumer's Surplus.
5. Evaluate $\int_1^5 \frac{dx}{\sqrt{(2x-1)}}$

6. If the marginal revenue is given by
   $MR = 27 - 12x + x^2$
   find the total revenue function.

1.6.9.2 Long Questions

1. The demand and supply functions under pure competition are $P = 1600 - x^2$ and $P = 2x^2 + 400$ respectively. Find the consumer's and producer's surplus.
2. Under a monopoly the quantity sold and market price are determined by the demand function. If the demand function for a profit maximization monopolist is $P = 274 - x^2$ and $MC = 4 + 3x$, find CS.
3. The demand function for a commodity $P = 30 - 2D$. The supply function $P = 3D$. Find Consumer's Surplus.
4. If the demand function is $P = 25 - 3x - 3x^2$ and the demand x_0 is 2, what will be the Consumer's Surplus ?
5. Given the Demand function $P = 8 - 2x$ and the supply function $P = 2 + x$, find the Consumer's Surplus and the Producer's Surplus.

   This lesson has been written from the assistance taken from Distance Education Council.

LESSON NO. 1.7 **AUTHOR : DR. VIPLA CHOPRA**

MAXIMA AND MINIMA (ONE VARIABLES)

1.7.0 Introduction:

We have studied the technique of derivatives in case of simple functions and logarithmic and exponential functions. In the present lesson for finding out the maximum and minimum values of a function the technique of derivatives have been used. The concept of the maxima and minima of functions helps a great deal in the study of problems of economic theory. It provides ways for a manufacturer or producer to maximise his profits or minimise his costs.

1.7.1 Objectives

After going through this lesson you will be able to

* distinguish between increasing and decreasing functions.
* differentiate between convexity and concavity of curves.
* learn about the conditions for a function (one variable) to be maximum/ minimum.

* know about the conditions for a function (two variables) to be maximum/ minimum.
* explain constrained Maxima and Minima with two variables.

1.7.2 Basic Concepts:

For finding out the maximum and minimum values of a function, two basic concepts namely the concepts of increasing and decreasing functions, and convexity and concavity of curves have been used. The technique of Maxima and Minima is quite beneficial in Economic Theory. For finding out maximum revenue, maximum profit, minimum cost, minimum average variable cost, etc. we use the technique of maxima and minima. Every consumer wants to maximise his utility. Every producer wants to maximise his revenue and wants to minimise his costs. All such problems can be solved through the technique of maxima and minima.

1.7.2.1 Increasing and Decreasing Functions:

A function y = f (x) is said to be an increasing function, if as x increases y also increases and the function is said to be decreasing function, if as x increaes y decreases.

Alternatively:

If y = f (x) then y is said to be an increasing $(\uparrow)$ function of x at the point x = a if $\left[\frac{dy}{dx} \text{at } x = a\right] > 0$ i.e. $\left(\frac{dy}{dx}\right) > 0$ and it is said to decreasing $(\downarrow)$ function of x at the point x = a, if $\left[\frac{dy}{dx} \text{at } x = a\right] < 0$ i.e. $\left(\frac{dy}{dx}\right) < 0.$

Graphically:

Let P be the price and q is the quantity supplied and suppose the relation between q and P as $P = q^2 + 2q + 1$. Now price increase as supply increases (See fig. 1) and

price increases as demand decreases (See Fig. 2).

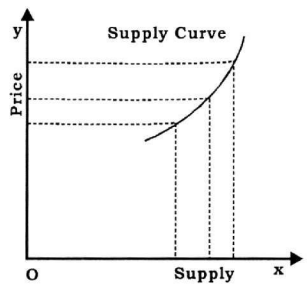


Fig. 1

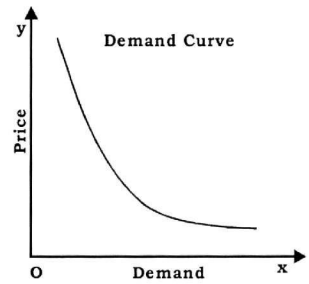


Fig. 2

Test for Increasing and Decreasing Functions:

(a) Let y = f (x) be an increasing function of x. As x increases from OM to ON, y increases from OM_1 to ON_1.

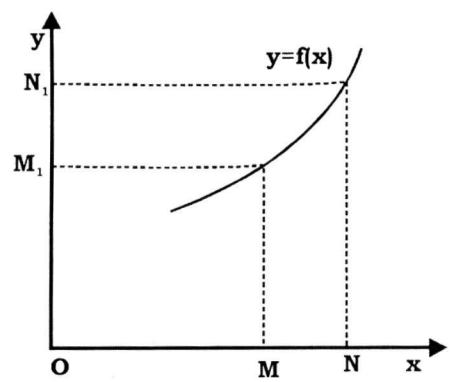


Fig. 3

$\therefore$ $ON_1 > OM_1$

$$f(x+\Delta x) > f(x)$$

$$f(x+\Delta x) - f(x) > 0$$

Let $\Delta x > 0$, divide both sides by Δx,

$$\therefore \quad \frac{f(x+\Delta x) - f(x)}{\Delta x} > 0$$

Take limits as $\Delta X \to 0$

$$\underset{\Delta x \to 0}{\text{Lt}} \frac{f(x+\Delta x)-f(x)}{\Delta x} > 0$$

$\therefore$ $f'(x) > 0$ or $\frac{d}{dx} f(x) > 0$ or $\frac{dy}{dx} > 0$

(b) Let y = f (x) be a decreasing function of x. As x increases from OM to ON, y increases from OM_1 to ON_1.

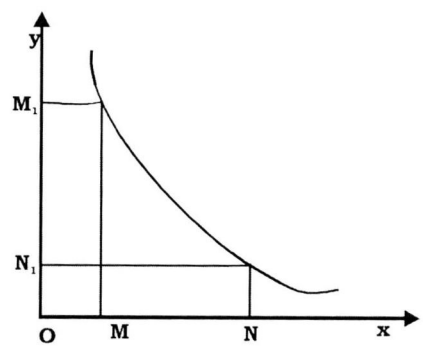


Fig. 4

$\therefore$ $ON_1 < OM_1$

or $f(x+\Delta x) < f(x)$

or $f(x+\Delta x) - f(x) < 0$

or Let $\Delta x > 0$, divide both sides by Δx

or $\frac{f(x+\Delta x)-f(x)}{\Delta x} < 0$

or $\underset{\Delta x \to 0}{\text{Lt}} \frac{f(x+\Delta x)-f(x)}{\Delta x} < 0$

or $\frac{d}{dx} f(x) < 0$ or $\frac{dy}{dx} < 0$

Thus I : If y = f (x) is an increasing function of x, then $\frac{dy}{dx}$ is positive.

II : If y = f (x) is a decreasing function of x, then $\frac{dy}{dx}$ is negative.

Example 1:

Test the cost function 120 – 16x + x^2 for increasing $(\uparrow)$ or decreasing $(\downarrow)$ function at the point (i) x = 4, (ii) x = 10.

Solution:

Since y = 120 – 16x + z .....(2)

$$\frac{dy}{dx} = -16 + 2x$$

When x = 4

$$\therefore \quad \frac{dy}{dx} = -16 + 2 \times 4 = -8 < 0$$

Therefore, at x = 4 the function is decreasing.

(ii) When x = 10

$$\frac{dy}{dx} = -16 + 2x = -16 + 2 \times 10 = 4 > 0$$

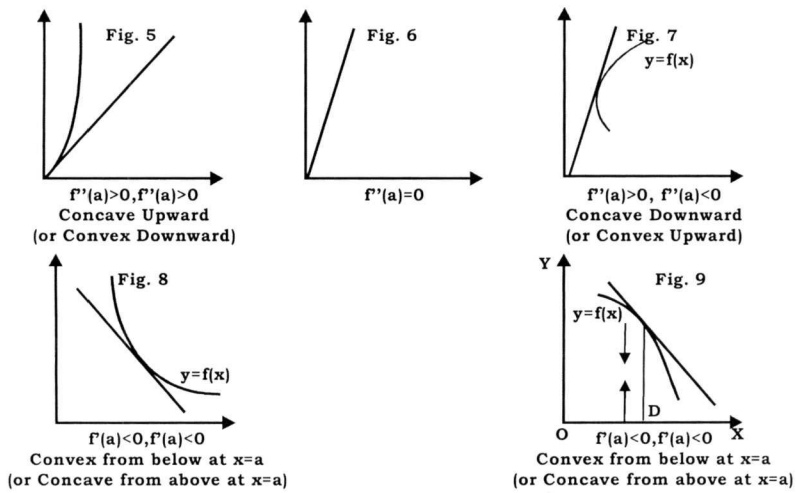
Thus at x = 10, $\frac{dy}{dx}$ is positive and therefore, the function is increasing at x = 10.

1.7.2.2 Convexity or Concavity of Curves:

In order to determine the concavity and covexity of the curve y = f (x), we consider the derivative of the second order. If f (x) > 0 then y has been defined as an increasing function of x. Three possible cases may be:

1. f''(x) > 0, then we say that the function is increasing at an increasing rate i.e. the rate of change of y is increasing. The curve lies above the tangent and we say that the curve is concave upwards (or convex downwards) (See Fig. 5).
2. f''(x) = 0, there will be no curvature and the curve will be a straight line (See Fig. 6).
3. If f''(x) < 0, then the curve lies below the tangent and we say that the curve is concave downward (or convex upward) (See Fig. 7).

Diagramatically:



If y = f (x), $\frac{dy}{dx} < 0$, then y has been defined as decreasing function of x.

4. f''(x) > 0. The curve will have shape as given in Fig. 8 above. It is concave upward or convex downward.
5. f''(x) < 0. The curve will have shape as given in Fig. 9. It is a position to decide about rising and falling nature of the curve.

Example 2:

Show that the demand curve $y = \frac{a}{x+b} - C$ is downward sloping and convex from below, a and b being positive constants.

Solution:

$$y = \frac{a}{x+b} - C,$$

$$\frac{dy}{dx} = -a(x+b)^{-2} = -\frac{a}{(x+b)^2}$$

$\therefore$ which is negative.

Hence, the given demand curve slopes downwards.

Now $\frac{d^2y}{dx^2} = \frac{2a}{(x+b)^3}$ Which is positive

$\therefore$ The given demand curve is convex downward or convex from below.

Example 3:

Show that the curve $y = \sqrt{a - bx} + c$ is downward sloping and convex from above, given that a, b and c are positive constants.

Solution:

$$y = \sqrt{a - bx} + c$$

$$\frac{1}{2}(a - bx)^{-1/2}(-b) = \frac{-b}{2}\left(\frac{1}{\sqrt{a - bx}}\right)$$

which is negative. Hence, the given demand curve slopes downwards.

Now $\frac{d^2y}{dx^2} = \frac{b}{4}(a - bx)^{-3/2}(-b) = -\frac{b^2}{4}(a - bx)^{-3/2} < 0$

$\therefore$ The given demand curve is convex upward.

1.7.3 Maximum and Minimum Values of a Function (One Variable):

1.7.3.1 Definition:

The function y = f (x) said to have maximum value at x = a if the value of f (x) at x = a is greater than all values in the immediate neighbourhood of x = a.

In Fig. 10 the value of y or f (x) increases as x increases upto x = a and then decreases. The value of f (x) at x = a is ordinate of E.

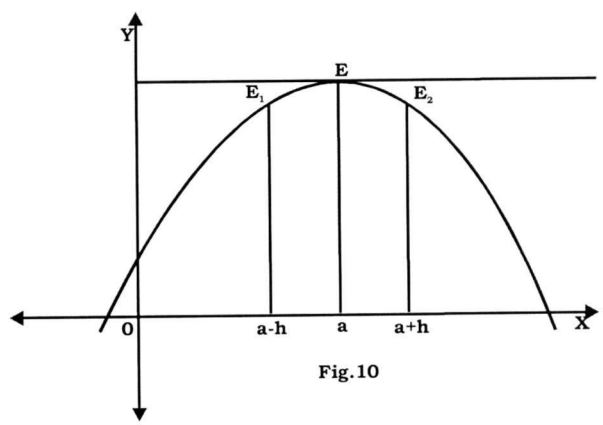


Fig.10

At point E the value of f(x) is greaer than all values in the immediate neighbourhood of E. Thus the function has maximum value at x = a.

i.e. f (a) > f (a + h), where a + h is a point to the right of x = a and also f (a) > f (a – h), where a – h is a point to the left of x = a. Similarly, f (x) is said to

have minimum value at x = a if the value of f (x) at x = a is smaller than all the values in the immediate neighbourhood of x = a.

In Fig. 11, the value of y or f (x) decreases upto x = a. At x = a, the value of y becomes stationary and then its value increases as x increases beyond x = a i.e. f (a) < f (a + h), where a + h is a point to the right of x = a and also f (a) < f (a – h), where a – h is a point to the left of x = a.

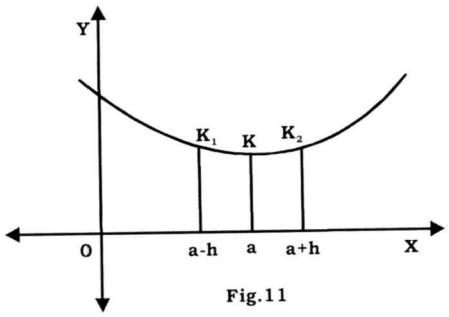


Fig.11

Such maximum and minimum values are known as extreme values of function.

1.7.3.2 Necessary and Sufficient Conditions:

(a) The necessary condition for all extreme values is that $\frac{dy}{dx}=0$ Fig. 10 and Fig. 11 show that maximum or minimum occurs at x = a where $\frac{dy}{dx}=0$. In Fig. 12, $\frac{dy}{dx}=0$ at point D on the curve y = f (x). Here the value is neither a maximum or minimum. Therefore, $\frac{dy}{dx}=0$ is a necessary condition but not a sufficient condition.

(b) (i) If f'(x) changes signs from positive to negative as x passes through the

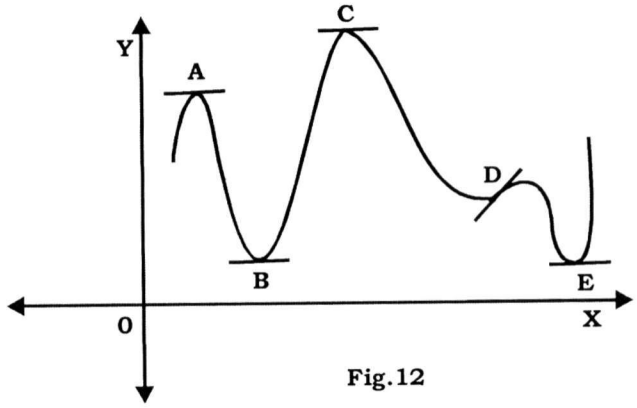


Fig.12

value a, where f(a) = 0, then f (a) will have a maximum value of f(x) at x = a.

(ii) If f'(x) changes signs from negative to positive as x passes though the value a, where f'(a) = 0, then f (a) is a minimum value of f(x) at x = a. The above condition taken together constitute conditions.

(iii) **Point of Inflexion:** If f'(x) does not change signs as x passes through the value a, where f'(a) = 0, then f(a) is neither a maximum or a minimum value of f'(x) at x = a is called a point of inflextion on the curve y = f(x).

1.7.3.3 Stationary Values or Turning Values:

If f'(x), the derivative of f(x) becomes zero at x = a, i.e. if f'(a) = 0, then the function is said to the stationary x = a and then f(a) is called the staitonary value or turning value of f(x). The reason of its being called stationary is that the rate of change of f(x) with respect to x is zero for those values of x at which f(x) is stationary.

Note:

It is clear that the maximum and minimum values are stationary but it is not necessary that every stationary value be either a maximum or minimum.

1.7.3.4 Properties of Maxima and Minima:

If f'(x) is a continuous function, its graph will be an unbroken curve and therefore, the adjoining figure will at once show that:

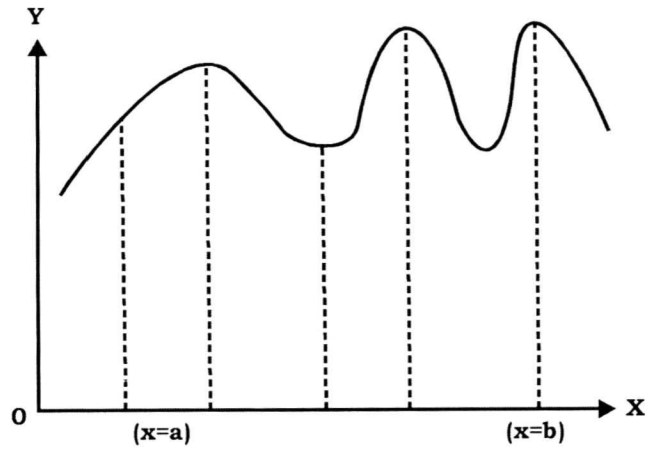


Fig.13

1. Between two equal values of a function at least one maximum or one minimum lie.
2. Maxima and Minima occur alternatively.
3. It is clear that the sign of $\frac{dy}{dx}$ changes from positive to negative as x passes (while increasing) through the value which makes y a maximum and it changes from negative to positive as x passes through the value which makes y a minimum.

Example 4:

Show that $y = x^2 - 4x$ is decreasing at $x = -1$, stationary at $x = 2$ and increasing at $x = 3$.

Solution:

Now $f(x) = x^2 - 4x$, diff. w.r. to x

$\therefore$ $f'(x) = 2x - 4$

At $x = -1$, $f'(-1) = 2(-1) - 4 = -6 =$ -ve

At $x = 2$, $f'(2) = 2(2) - 4 = 4 - 4 = 0$

At $x = 3$, $f'(3) = 2(3) - 4 = 6 - 4 = 2 =$ +ve

We know that $y = f(x)$ is increasing function if $f'(x)$ is positive, decreasing function if $f'(x)$ is negative and stationary if $f'(x) = 0$.

At $x = -1$, $f'(x)$ is -ve, so function is decreasing.

At $x = 3$, $f'(x)$ is +ve, so function is increasing.

At $x = 2$, $f'(x) = 0$, so function is stationary.

Example 5:

Investigate the maxima and the minima of the ordinate of the curve $y = (x - 2)^6 (x - 3)^5$

Solution:

$y = (x - 2)^6 (x - 3)^5$

Take log on both sides, we have

$\log y = 6 \log (x - 2) + 5 \log (x - 3)$

Now differentiate w.r. to x

$$\frac{1}{y}\frac{dy}{dx} = \frac{6}{x-2} + \frac{5}{x-3}$$

$$= \frac{dy}{dx} = (x-2)^6 (x-3)^5 \left[\frac{6}{x-2} + \frac{5}{x-3}\right]$$

$$\frac{dy}{dx} = (x-2)^5 (x-3)^4 (11x-28)$$

Hence for a max. or min. y,

$$\frac{dy}{dx} = 0, \text{ i.e. } x = 2, 3, \frac{28}{11}$$

Now if x be a little than 2, $\frac{dy}{dx}$ (-ve) (+ve) (-ve) = +ve and if x be a little greater than 2, $\frac{dy}{dx}$ = (+ve) (+ve) (-ve) = -ve

Hence there is a change in the sign of $\frac{dy}{dx}$ viz., from positive to negatie as x passes through the value 2, so that y assumes a maximum value at x = 2.

Example 6:

Examine the function $\frac{(x+2)(x+3)}{(x-2)(x-3)}$ for maximum and minimum values.

Solution:

Let $y = \frac{(x+2)(x+3)}{(x-2)(x-3)} = \frac{x^2+5x+6}{x^2-5x+6} = 1 + \frac{10x}{x^2-5x+6}$

or $$y = 1 + \frac{10x}{(x-2)(x-3)}$$

$$\frac{dy}{dx} = \frac{(x^2 - 5x + 6)10 - 10x(2x-5)}{(x-2)^2\,(x-3)^2} = \frac{-10x^2 + 60}{(x-2)^2\,(x-3)^2} = \frac{-10\,(6 - x^2)}{(x-2)(x-3)^2}$$

$$=> x = \pm\sqrt{6}$$

$$\frac{d^2y}{dx^2} = \frac{10(x-2)^2\,(x-3)^2\,(-2x) - (6 - x^2)\frac{d}{dx}\left[(x-2)^2\,(x-3)^2\right]}{(x-2)^4\,(x-3)^4}$$

At $x = \sqrt{6}, \frac{d^2y}{dx^2}$ is negative, $\therefore$ y is minimum

$$y\ \min = \frac{(\sqrt{6}+2)(\sqrt{6}+3)}{(\sqrt{6}-2)(\sqrt{6}-3)} = -(49 - 20\sqrt{6})$$

Example 7:

Find the maximum or minimum values of $x^2 e^{1/x^2}$

Solution:

Let y = $x^2 e^{1/x^2}$

$$\therefore \quad \frac{dy}{dx} = x^2.e^{1/x^2}\,\frac{d}{dx}\left(\frac{1}{x^2}\right) + 2x.e^{1/x^2}$$

$$= x^2 e^{1/x^2}\left(-\frac{2}{x^3}\right) + 2xe^{1/x^2}$$

$$= -2e^{1/x^2}\ \frac{1}{x} + 2xe^{1/x^2} = 2e^{1/x^2}\left(x - \frac{1}{x}\right)$$

$$\frac{d^2y}{dx^2} = 2^{e\frac{1}{x^2}}\left(1 + \frac{1}{x^2}\right) + \left(1 + \frac{1}{x}\right).2e^{x\frac{1}{2}}\left(-2x^{-3}\right)$$

$$\frac{d^2y}{dx^2} = \left[2e^{1/x^2}\left(1 + \frac{1}{x^2}\right) + \left(x - \frac{1}{x}\right)2.e^{1/x^2}\left(-\frac{2}{x^3}\right)\right]$$

$$= 2e^{1/x^2}\left[1 + \frac{1}{x^2} - \frac{2}{x^2} + \frac{2}{x^4}\right]$$

$$= \frac{2}{2x^4}.e^{1/x^2}\left[x^4 - x^2 + 2\right]$$

For stationary values, $\frac{dy}{dx} = 0$, thus

$$2.e^{1/x^2}\left(x - \frac{1}{x}\right) = 0$$

$x^2 = 1$
$x = \pm 1$

For x = 1, $\frac{d^2y}{dx^2}$ is positive

Hence y is minimum at x = 1
and its minimum value = i.e. = e.

Example 8:

Find the maximum value of $\left(\frac{1}{x}\right)^x$

Solution:

Let $y = \left(\frac{1}{x}\right)^x = x^{-x}$

Take Log

$\therefore$ $\log y = -x \log x$

Differentiate w.r. to x

$$\frac{d}{dx}(\log y) = -\frac{d}{dx}(x \log x)$$

$$\frac{1}{y}.\frac{dy}{dx} = -\left(x\frac{d}{dx}\log x + \log x.\frac{d}{dx}.x\right)$$

$$= -\left(x.\frac{1}{x} + \log x.1\right)$$

$$\frac{1}{y}.\frac{dy}{dx} = -(1 + \log x)$$

$$\frac{dy}{dx} = -y\left(1 + \log x\right)$$

Differentiate again w.r. to x

$$\frac{d^2y}{dx^2} = -\left[y.\frac{d}{dx}\left(1 + \log x\right) + \left(1 + \log x\right)\frac{dy}{dx}\right]$$

$$= -\left(y.\frac{1}{x} + \left(1 + \log x\right)^2\left(-y\right)\right)$$

$$= -y\left(\frac{1}{x} - \left(1 + \log x\right)^2\right) \qquad \left[\frac{dy}{dx} = -y(1 + \log x)\right]$$

$\frac{dy}{dx} = 0$ gives 1 + log = 0 $\therefore$ log x = –1 = –log e

log x = $\log e^{-1}$ = > x = e^{-1}

At x = e^{-1}, $\frac{d^2y}{dx^2} < 0$ $\therefore$ x = e^{-1} gives max. values

1.7.4 Summary

In this lesson the technique of finding maxima and minima in case of one variable function and two variable functions has been discussed. Two basic concepts namely increasing and decreasing functions and convexity and concavity of curves used in this technique have been explained. Both necessary and sufficient conditions for a function (one variable/two variable) to be maxima/minima have been stated. Each secion in this lesson is followed by unsolved exercise for practice to the reader.

1.7.5 Key Words

Local maximum : A point on a curve that is highest than the points on both sides of itself. A point where $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} <$

Local Minimum : A point on a curve that is lower than the points on both sides of itself. A point where $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} > 0$

Point of inflection : A point on a curve at which $\frac{dy}{dx}$ may or may not be zero, $\frac{d^2y}{dx^2} = 0$

Tangent : A straight line that touches a non-linear function at only one point, not cutting through the curve at the point. The slope of the tangent is used as a measure of the slope of the curve at that point.

1.7.6 Suggested Readings :

1. C.S. Aggarwal and R.C. Joshi : *Mathematics for Students of Economics.*
2. O.P. Bhardwaj and J.R. Sabharwal : *Mathematics for Students of Economics.*
3. G.C. Sharma and Madhu Jain : *Quantitative Techniques for Management.*
4. B.C. Mehta and G.M.K. Madnani : *Mathematics for Economists.*
5. S.C. Aggarwal and R.K. Rana : *Basic Mathematics for Economists.*

1.7.7 List of Questions:

1.7.7.1 Short Questions :

(i) Define the following : (a) increasing function (b) decreasing function

(ii) What is point of inflexion ?

(iii) Explain the following :
(a) concave upword (b) concave downword

1.7.7.2 Long Questions :

(i) If the demand function is P = $\sqrt{9-0}$ find at what level of output Q, the total revenue TR, will be maximum and what will it be ?

(ii) Show that the curve $y = \frac{a}{x+b} - c$ and $y = (a - bx)^2$ are downword sloping and convex from below.

(iii) Prove that $x^5 - 5x^4 + 5x^3$ has a maximum for x = 1, maximum nor a minimum.