**B.A. PART-III** **MATHEMATICS : PAPER-III (OPT.1)**
**SEMESTER-V** **DISCRETE MATHEMATICS-I**

**Unit No. – 1 and 2**

**LESSON NO.**



**Note :** Students can download the syllabus from department's website www.pbidde.org

**Department of Distance Education**
**Punjabi University, Patiala**

**LESSON NO. 1.1** **Author : Dr. Chanchal**

# SET THEORY, INCLUSION-EXCLUSION PRINCIPLE LANGUAGES & GRAMMARS

**Structure :**



## 1.1.1 Introduction

Firstly, we introduce a set as :

**Def. Set :** A set is a well defined collection of distinct objects.

The word 'well defined' implies that we are given a rule with the help of which we can say whether a particular object belongs to the set or not. The word 'distinct' implies that repetition of objects is not allowed. Each object of the set is called an element of the set. Further, sets are generally denoted by capital letters A,B,C,..... while elements of the sets are denoted by small letters a,b,c, ........ .

**For Example :** (i) The set of days of a week.
(ii) The set of even integers.

A set can be represented in two ways :

(1) Tabular or Roster Method

(2) Set-builder or Rule Method

In roster form, we represent a set by listing all its elements within curly brackets { }, separated by comma's while in the set-builder form, we do not list the elements but the set is represented by specifying the defining property.

| **For Example :** Set | Roster form | Set-builder form |
| --- | --- | --- |
| (1) A set of vowels | A = {a, e, i, o, u} | A = {x : x is a vowel of english alphabet} |
| (2) A set of positive even integers upto 10 | A = {2, 4, 6, 8, 10} | A = {x : x is a positive even integer and $x \leq 10$} |

There are some basic mathematical sets, such as

**N =** Set of all natural numbers

**W =** Set of all whole numbers

**I a Z =** Set of all integers

**Q =** Set of all rational numbers

**R =** Set of all real numbers

### 1.1.2 Some Basic Terms in Set Theory

**Membership of a Set :** If an object x is a member of the set A, we write $x \in A$, which can be read as 'x belongs to A' or A contains x. Similarly, we write $x \notin A$ to show that x is not a member of the set A.

**For Example :** Let A = {1, 2, 4, 6, 7}. Here $2 \in A$ but $5 \notin A$.

**Finite Set :** A set is said to be finite if it has finite no. of elements.

**For Example :** A = {2, 4, 6, 8}

**Infinite Set :** A set is said to be infinite if it has an infinite number of elements.

**For Example :** A = {1,2, 3.......} and B = {x : x is an odd integer} are infinite sets

**Singleton Set :** A set containing only one element is called a singleton or a unit set.

**For Example :** A = {x : x is a perfect square and $30 \leq x \leq 40$} = {36}

**Empty, Null or Void Set :** A set which contains no element, is called a null set and is denoted by $\phi$ (read as phi).

**For Example :** $A = \left\{ x : x \text{ is a positive integer satisfying } x^2 = \frac{1}{4} \right\}$

**Sub-Set, Super-Set :** If every member of a set A is a member of a set B, then A is called sub-set of B and B is called super-set of A.

or if $x \in A \Rightarrow x \in B$, then A is a sub set of B and B is a super set of A and we write it as, $A \subset B$ which means A is contained in B or B contains A.

**Note 1.** Since every element of A belongs to A

$\therefore$ $A \subseteq A \Rightarrow$ every set is sub set of itself.

**2.** The empty set $\phi$ is taken as a sub-set of every set.

**For Example :** Let A = {1, 2, 3, 4, 5, 6, 8, 10}, B = {2, 4, 6, 10}, C = {1, 2, 7, 8}.

Now every element of B is an element of A, $\therefore$ $B \subset A$

Again $7 \in C$, but $7 \notin A$

$\therefore$ $C \not\subset A$ i.e., C is not a sub-set of A.

**Equality of Sets :** Two sets A and B are said to be equal if both have the same elements. In other words, two sets A and B are equal when every element of A is an element of B and every element of B is element of A.

i.e., if $A \subset B$ and $B \subset A$, then A = B.

**For Example :** A = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10} and B = {x : x is a natural number and $1 \leq x \leq 10$}

Hence, A = B.

**Proper Sub-set :** A non-empty set A is said to be a proper subset of B if $A \subset B$ and $A \neq B$.

**Note : (i)** $\phi$ and A are called improper subsets of A.

**(ii)** If A has n elements, then number of subsets of A is $2^n$.

**Power set :** The power set of a finite set is the set of all sub-sets of the given set. Power set of A is denoted by P(A).

**For Example :** Take A = {1, 2, 3}

$\therefore$ P(A) = {$\phi$, {1}, {2}, {3}, {1, 2}, {1, 3}, {2, 3}, {1, 2, 3}}

**Universal Set :** If all the sets under consideration are sub-sets of a fixed set U, then U is called a universal set.

**For Example :** In Plane geometry, the universal set consists of points in a plane.

**Comparable and Non-comparable Sets :** Two sets are said to be comparable if one of the two sets is a sub-set of the other. Otherwise they are said to be non-comparable.

**For Example :** Let A = {2, 3, 5}, B = {2, 3, 5, 6}.

Here $A \subset B$, so A and B are comparable sets.

**Order of a Finite Set :** The number of different elements of a finite set A is called the order of A and is denoted by O(A).

**For Example :** If A = {2, 3, 6, 8}, then O(A) = 4

**Equivalent Sets :** Two finite sets A and B are said to be equivalent sets if the total number of elements in A is equal to the total number of elements in B.

**For Example :** Let A = {1, 2, 3, 4, 6}, B = {1, 2, 7, 9, 12}

$\therefore$ O(A) = 5 = O(B) $\Rightarrow$ A and B are equivalent sets, denoted as A ~ B.

**Cardinality :** Number of different elements in a set is known as its cardinality.

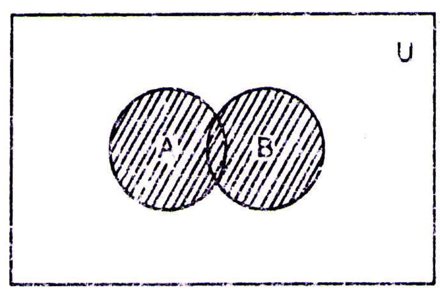
### 1.1.3 Operations on Sets

In order to represent various operations on sets, we use a special type of diagrams, called venn diagrams defined as :

**Venn Diagrams :** The relations between sets can be illustrated by certain diagrams called **Venn diagrams.** In a Venn diagram, universal set U is represented by a rectangle and any sub-set of U is represented by a circle within a rectangle U.
Now, various operations of set theory are discussed below :

#### 1.1.3.1 Union of Two Sets

If A and B be two given sets, then their union is the set consisting of all the elements of A together with all the elements in B. We should not repeat the elements. The union of two sets A and B is written as $A \cup B$.
In symbols, $A \cup B = \{x : x \in A \text{ or } x \in B\}$



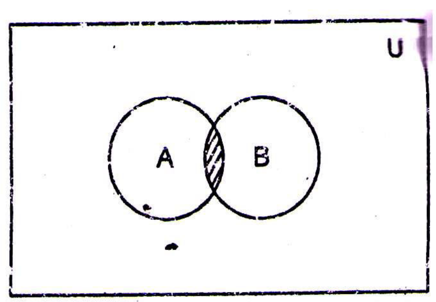

A ∪ B IS SHIELDED LIKE

**For Example :** Let A = {1, 2, 3, 5, 8}, B = {2, 4, 6}

$\therefore \quad A \cup B = \{1, 2, 3, 4, 5, 6, 8\}$

#### 1.1.3.2 Intersection of two sets

The intersection of two sets A and B, denoted by $A \cap B$, is the set of elements common to A and B.
In symbols, $A \cap B = \{x : x \in A \text{ and } x \in B\}$




A ∩ B IS SHIELDED LIKE

**For Example :** Let A = {2, 4, 6, 8, 10, 12}, B = {2, 3, 5, 7, 11}

$\therefore \quad A \cap B = \{2\}$

**Disjoint Sets :** If A and B are two given sets such that $A \cap B = \phi$, then the sets A and B are said to be disjoint.

**For Example :** Let A = {a, b, c, d}, B = {l, m, n, p},

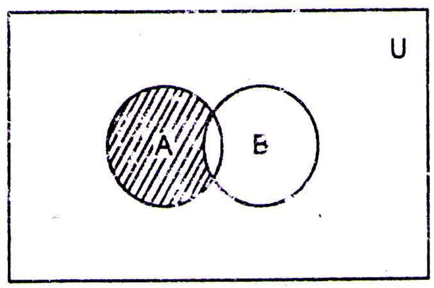
$\therefore \quad A \cap B = \phi \Rightarrow$ A and B are disjoint sets.

### 1.1.3.3 Difference of two sets

The difference of two sets A and B is the set of those elements of A which do not belong to B. We denote this by A – B.

In symbols, we write $A - B = \{x : x \in A \text{ and } x \notin B\}$

A – B is also sometimes written as A/B.




A – B IS SHIELDED LIKE

[Similarly, We can define $B\text{-}A = \{x : x \in B \text{ and } x \notin A\}$]

**For Example :** Let A = {a, b, c, d, e}, B = {c, d, e, f, g}

Then, A – B = {a, b} and B-A = {f, g}

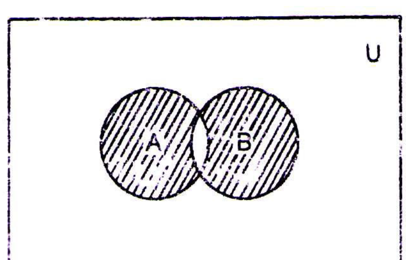
**Note.** $B - A \neq A - B$

### Symmetric Difference of Two Sets

If A and B are any two sets, then the set $(A - B) \cup (B - A)$ is called symmetric difference of A and B and is denoted by $A \Delta B$.

In symbols, we write

$$A \Delta B = \{x : (x \in A \text{ and } x \notin B) \text{ or } (x \in B \text{ and } x \notin A)\}$$



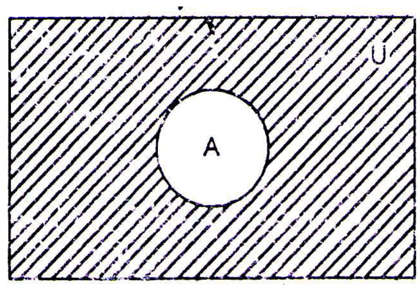

A Δ B IS SHIELDED LIKE

**For Example :** Let A = {1, 2, 4}, B = {1, 2, 3, 4, 6}

$\therefore$ $A \Delta B = (A/B) \cup (B/A) = (A - B) \cup (B - A) = \{4\} \cup \{3, 5, 6\} = \{3, 4, 5, 6\}$.

### 1.1.3.4 Complement of a Set

Let A be a subset of universal set U. Then the complement of A is the set of all those elements of U which do not belong to A and we denote complement of A by $A^c$ or $A'$. We can write $A^c = \{x : x \in U, x \notin A\}$




$A^c$ IS SHIELDED LIKE

**For Example :** If U = {2, 4, 6, 8,10}, A = {4, 8}, then $A^c = \{2, 6, 10\}$

**Note.** $U^c = \phi$ and $\phi^c = U$, $(A^c)^c = A$.

### 1.1.4 Some Fundamental Laws of Algebra of Sets

### I. Idempotent Laws

**Statement :** If A is any set, then (i) $A \cup A = A$ (ii) $A \cap A = A$

**Proof :** (i) L.H.S. $= A \cup A$

$= \{x : x \in A \cup A\} = \{ x : x \in A \text{ or } x \in A\}$

$= \{x : x \in A\} = A$

= R.H.S.

(ii) Do Yourself.

### II. Identity Laws

**Statement.** If A is any set, then (i) $A \cup \phi = A$ (ii) $A \cap U = A$

**Proof :** (i) L.H.S. $= A \cup \phi = \{x : x \in A \cup \phi\}$

$= \{x : x \in A \text{ or } x \in \phi\} = \{x : x \in A\}$

= A = R.H.S.

(ii) Do Yourself.

### III. Commutative Laws

**Statement.** If A and B are any two sets, then (i) $A \cup B = B \cup A$ (ii) $A \cap B = B \cap A$

**Proof :** Do Yourself.

## IV. Associative Laws

**Statement.** If A, B and C are any three sets, then

(i) $A \cup (B \cup C) = (A \cup B) \cup C$ (ii) $A \cap (B \cap C) = (A \cap B) \cap C$

**Proof :** Do Yourself.

## V. Distributive Laws

**Statement.** If A, B, C are any three sets, then

(i) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

(ii) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

**Proof :** L.H.S. $= A \cup (B \cap C)$

$= \{x : x \in A \cup (B \cap C)\}$

$= \{x : x \in A \text{ or } x \in (B \cap C)\}$

$= \{x : x \in A \text{ or } (x \in B \text{ and } x \in C)\}$

$= \{x : (x \in A \text{ or } x \in B) \text{ and } (x \in A \text{ or } x \in C)\}$

$= \{x : x \in (A \cup B) \text{ and } x \in (A \cup C)\}$

$= \{x : x \in (A \cup B) \cap (A \cup C)\}$

$= \{(A \cup B) \cap (A \cup C)\}$

= R.H.S.

$\therefore \quad A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

**Note.** We can also prove above result by showing that

$A \cup (B \cap C) \subset (A \cup B) \cap (A \cup C)$ and $(A \cup B) \cap (A \cup C) \subset A \cup (B \cap C)$

(ii) Do Yourself.

## VI. De Morgan's Laws

**Statement.** If A and B are two sub-sets of U, then

(i) $(A \cup B)^c = A^C \cap B^c$ (ii) $(A \cap B)^c = A^c \cup B^c$

**Proof :** (i) L.H.S. $= (A \cup B)^c = \{x : x \in (A \cup B)^c\}$

$= \{x : x \notin (A \cup B)\}$

$= \{x : x \notin A \text{ and } x \notin B\}$

$= \{x : x \in A^c \text{ and } x \in B^c\}$

$= \{x : x \in (A^c \cap B^c)\}$

$= A^c \cap B^c =$ R.H.S.

$\therefore \quad (A \cup B)^c = A^c \cap B^c$

(ii) Do Yourself.

## 1.1.5 Some Important Examples

**Example 1.1 :** Let U be the set of integers and let A = {x : x is divisible by 3}, let B = {x : x is divisible by 2}. Let C = {x : x is divisible by 5}. Find the elements in each of the following set :

(a) $A \cap B$ (b) $A \cup C$ (c) $A \cap (B \cup C)$ (d) $(A \cap B) \cup C$
(e) $A^c \cap B^c$ (f) $(A \cap B)^c$ (g) $B/A$ (h) $A/B$ (i) $A/(B/C)$

**Sol. :** $U = \{0, \pm 1, \pm 2, \pm 3, \pm 4, \ldots\ldots\ldots\}$

$A = \{x : x \text{ is divisible by } 3\} = \{x : x = 3n, n \in I\} = N_3$

$B = \{x : x \text{ is divisible by } 2\} = \{x : x = 2n, n \in I\} = N_2$

$C = \{x : x \text{ is divisible by } 5\} = \{x : x = 5n, n \in I\} = N_5$

(a) $A \cap B = N_3 \cap N_2 = N_6 \quad \because$ l.c.m. $\{2, 3\} = 6$

$= \{\ldots\ldots\ldots\ -12, -6, 0, 6, 12, \ldots\ldots\}$

(b) $A \cup C = N_3 \cup N_5 = \{\ldots\ldots\ldots\ -9, -6, -5, -3, 0, 3, 5, 6, \ldots\ldots\}$

(c) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C) = (N_3 \cap N_2) \cup (N_3 \cap N_5) = N_6 \cup N_{15}$

$= \{\ldots\ldots -15, -12, -6, 0, 6, 12, 15 \ldots\ldots\ldots\}$

(d) $(A \cap B) \cup C = \{N_3 \cap N_2\} \cup N_5 = N_6 \cup N_5$

$= \{\ldots\ldots\ -12 - 10, 6, -5, 0, 5, 6, 10, 12, \ldots\ldots\}$

(e) $A^c \cap B^c = (A \cup B)^c = (N_3 \cup N_2)^c$

$= \{\ldots\ldots\ -11, -7, -5, -1, 1, 5, 7, 11, \ldots\ldots\ldots\}$

(f) $(A \cap B)^c = (N_3 \cap N_2)^c = (N_6)^c$

$= \{-7, -5, -3, -2, -1, 1, 2, 3, 4, 5, \ldots\ldots\ldots\}$

(g) $A/B = N_3/N_2 = N_3 - N_2$

$= \{\ldots\ldots\ -10, -8, -4, -2, 2, 4, 8, 10 \ldots\ldots\}$

(h) $B/A = N_2/N_3 = N_2 - N_3$

$= \{\ldots\ldots\ldots\ -15, -9, -3, 3, 9, 15, \ldots\ldots\}$

(f) $A/(B/C) = (A/B) \cup (A \cap C)$

$= (N_3/N_2) \cup (N_3 \cap N_5) = (N_3/N_2) \cup N_{15}$

$= \{\ldots\ldots\ldots, -15, -9, -3, 3, 9, 15, \ldots\ldots\ldots\} \cup \{\ldots\ldots\ 30, 15, 0, 15, 30 \ldots\ldots\}$

$= \{\ldots\ldots\ldots, -15, -9, -3, 3, 9, 15, \ldots\ldots\ldots)$.

**Example 1.2 :** Prove that $A \cup (B/A) = A \cup B$.

**Sol. :** L.H.S. $= A \cup (B/A) = A \cup (B - A)$

$= A \cup (B \cap A^c)$ $[A - B = A \cap B^c]$

$= (A \cup B) \cap (A \cup A^c)$ [Distributive Law]

$= (A \cup B) \cap X$ $[A \cup A^c = X]$

$= A \cup B =$ R.H.S.

**Example 1.3 :** Let $A = \{1, 2, 4\}$, $B = (4, 5, 6\}$,

Find $A \cup B$, $A \cap B$ and $A - B$.

**Sol.** $A = \{1, 2, 4\}$ and $B = \{4, 5, 6\}$

(i) $A \cup B = \{1, 2, 4\} \cup \{4, 5, 6\} = \{1, 2, 4, 5, 6\}$

(ii) $A \cap B = \{1, 2, 4\} \cap \{4, 5, 6\} = \{4\}$

(iii) $A - B = \{1, 2, 4\} - (4, 5, 6\} = \{1, 2\}$.

**Example 1.4 :** Prove that $A \cup B = A \cap B$ iff $A = B$

**Sol. :** (i) Assume that $A \cup B = A \cap B$ ... (1)

Let x be any element of A

$\therefore$ $x \in A \Rightarrow x \in A \cup B \Rightarrow x \in A \cap B$ [$\because$ of (1)]

$\Rightarrow x \in B$

$\therefore$ $x \in A \Rightarrow x \in B$

$\therefore$ $A \subset B$

Similarly, $B \subset A$ ... (2)

From (2) and (3), $A = B$. ... (3)

$\therefore$ $A \cup B = A \cap B \Rightarrow A = B$

(ii) Assume that $A = B$

$\therefore$ $A \cup B = A \cup A = A$

$A \cap B = A \cap A = A$

$\therefore$ $A \cup B = A \cap B$

$\therefore$ $A = B \Rightarrow A \cup B = A \cap B$.

**Example 1.5 :** For any two sets A and B, prove that $A \cap B = \phi \Rightarrow A \subset B^c$.

**Sol. :** We are given that $A \cap B = \phi$ ... (1)

Let x be any element of A

$\therefore$ $x \in A \Rightarrow x \notin B$ [$\because$ of (1)]

$\Rightarrow$ $x \in B^c$

$\therefore$ $x \in A \Rightarrow x \in B^c$

But x is any element of A

$\therefore$ every element of A is an element of $B^c$

$\therefore$ $A \subset B^c$.

**Example 1.6 :** Prove that $A^c/B^c = B/A$

**Sol. :** L.H.S. $= A^c/B^c$

$= A^c - B^c = \{x : x \in (A^c - B^c)\} = \{x : x \in A^c \text{ and } x \notin B^c\}$

$= \{x : x \notin A \text{ and } x \in B\} = \{x : x \in B \text{ and } x \notin A\}$

$= \{ x : x \in (B - A)\} = B - A = B/A$

= R.H.S.

$\therefore \quad A^c/B^c = B/A.$

**Example 1.7 :** Show that $A \cap (B - C) = (A \cap B) - (A \cap C)$

**Sol. :** R.H.S. $= (A \cap B) - (A \cap C)$

$= (A \cap B) \cap (A \cap C)^c = (A \cap B) \cap (A^c \cup C^c)$

$= [(A \cap B) \cap A^c] \cup [A \cap B) \cap C^c]$

$= [(A \cap A^c) \cap B] \cup [A \cap (B \cap C^c)] = (\phi \cap B) \cup [A \cap (B - C)]$

$= \phi \cup [A \cap (B - C)] = A \cap (B - C) = $ L.H.S.

## 1.1.6 Cartesian Product of Sets

**Ordered-Pair :** By an ordered pair of elements, we mean a pair (a, b) such that $a \in A$ and $b \in B$. The ordered pairs (a, b), (b, a) are different unless a = b. Also} (a, b) = (c, d) iff a = c, b = d.

**Cartesian Product of Two Sets :** The set of all ordered pairs (a, b) of element $a \in A$, $b \in B$ is called the cartesian product of the sets A and B and is denoted by $A \times B$.

In symbols, $A \times B = \{(a, b) : a \in A, b \in B\}$

**Note 1.** $A \times B$ and $B \times A$ are different sets if $A \neq B$.

**2.** $A \times B = \phi$ when one or both of A, B are empty.

**Art 1.1 :** Prove that

(i) $A \times (B \cup C) = (A \times B) \cup (A \times C)$

(ii) $A \times (B \cap C) = (A \times B) \cap (A \times C)$

**Proof :** (i) L.H.S. $= A \times (B \cup C)$

$= \{(x, y) : x \in A \text{ and } y \in (B \cup C)\}$

$= \{(x, y) : x \in A \text{ and } (y \in B \text{ or } y \in C)\}$

$= \{(x, y) : (x \in A \text{ and } y \in B) \text{ or } (x \in A \text{ and } y \in C)\}$

$= \{(x, y) : (x, y) \in (A \times B) \text{ or } (x, y) \in (A \times C)\}$

$= \{(x, y) : (x, y) \in (A \times B) \cup (A \times C)\}$

$= (A \times B) \cup (A \times C) = $ R.H.S.

$\therefore \quad A \times (B \cup C) = (A \times B) \cup (A \times C)$

(ii) L.H.S. $= A \times (B \cap C)$

$= \{(x, y) : x \in A \text{ and } y \in (B \cap C)\}$

$= \{(x, y) : x \in A \text{ and } (y \in B \text{ and } y \in C)\}$

$= \{(x, y) : (x \in A \text{ and } y \in B) \text{ and } (x \in A \text{ and } y \in C)\}$

$= \{(x, y) : (x, y) \in (A \times B) \text{ and } (x, y) \in (A \times C)\}$

$= \{(x, y) : (x, y) \in (A \times B) \cap (A \times C)\}$

$= (A \times B) \cap (A \times C) = \text{R.H.S}$

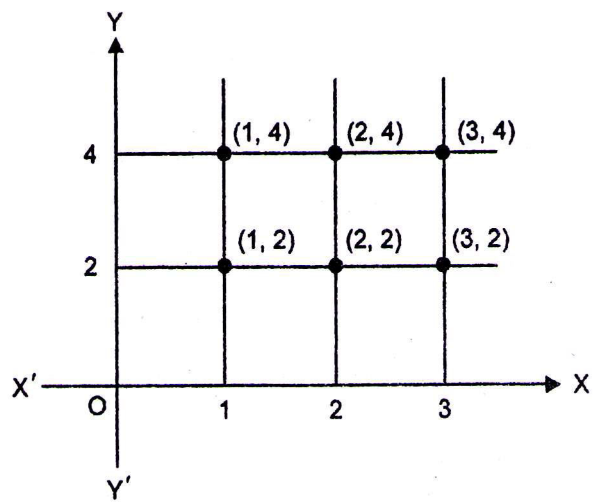
$\therefore \quad A \times (B \cap C) = (A \times B) \cap (A \times C).$

**Example 1.8 :** Let A = {1, 2, 3}, B = {2, 4}. Find A × B and show it graphically.

**Sol.** Here A = {1, 2, 3}, B = {2, 4}

$A \times B = \{1, 2, 3\} \times \{2, 4\} = \{(1, 2), (1, 4), (2, 2), (2,4), (3, 2), (3, 4)\}.$

Now to represent (1, 2), we draw a vertical line through 1 and a horizontal line through 2. These two lines meet in the point which represents (1, 2). Similarly we can represent the other points in A × B and get the graphical representation of A × B.




**Example 1.9 :** A, B, C are any three sets, then prove that

$(A \cap B) \times C = (A \times C) \cap (B \times C)$

**Sol. :** L.H.S. $= (A \cap B) \times C$

$= \{(x, y) : x \in (A \cap B) \text{ and } y \in C\}$

$= \{(x, y) : (x \in A \text{ and } x \in B) \text{ and } y \in C\}$

$= \{(x, y) : (x \in A \text{ and } y \in C) \text{ and } (x \in B \text{ and } y \in C)\}$

$= \{(x, y) : (x, y) \in (A \times C) \text{ and } (x, y) \in (B \times C)\}$

$= \{(x, y) : (x, y) \in (A \times C) \cap (B \times C)\}$

$= (A \times C) \cap (B \times C) = \text{R.H.S.}$

## 1.1.7 Partition of Sets

A partition of a non-empty set A is a collection $P = \{A_1, A_2, A_3, \ldots\ldots\ldots\ldots\}$ of subsets of A such that

(i) $A = A_1 \cup A_2 \cup A_3 \cup \ldots\ldots\ldots\ldots$

and (ii) $A_i \cap A_j = \phi$ for $i \neq j$

$A_1, A_2, A_3, \ldots\ldots$ are called cells or blocks of the partition P.

**For Example :** (i) Let $A = \{a, b, c\}$ be any set.

Then, $P_1 = \{\{a\}, \{b\}, \{c\}\}$, $P_2 = \{\{a\}, \{b, c\}\}$, $P_3 = \{\{b\}, \{a, c\}\}$,

$P_4 = \{\{c\}, \{a, b\}\}$, $P_5 = \{\{a, b, c\}\}$ are partitions of the set A.

(ii) Let **Z** = set of integers. Then the collection

$P = \{\{n\} : n \in \mathbf{Z}\}$ is a partition of **Z**.

**Minimum Set or Minset or Minterm :**

Let A be any non-empty set and $B_1, B_2, \ldots\ldots\ldots, B_n$ be any subsets of A. Then the minimum set generated by the collection $\{B_1, B_2, \ldots\ldots, B_n\}$ is a set of the type $D_1 \cap D_2 \cap \ldots\ldots\ldots \cap D_n$, where each $D_1, D_2, \ldots\ldots\ldots, D_n$ is $B_i$ or $B_i^c$ for $i = 1, 2, 3, \ldots\ldots, n$.

**For Example :** The minsets generated by two sets $B_1$ & $B_2$ are

$$A_1 = B_1 \cap B_2', \; A_2 = B_1 \cap B_2, \; A_3 = B_{1S}^{C} \cap B_2, \; A_4 = B_1^{C} \cap B_2^{C}.$$

**Normal form (or Canonical form) :**

A set F is said to be in minset normal (or cannonical) form when it is expressed as the union of distinct non-empty minsets or it is $\phi$

i.e., either $F = \phi$ or $F = \bigcup_{\lambda \in \Delta} A_i$, where $A_\lambda$ 's are non-empty minsets.

**Principle of Duality for Sets :**

Let S be any identity in set theory involving the operation union ($\cup$), intersection ($\cap$). Then the statement S* obtained from S by changing union to intersection to union and empty set $\phi$ to universal set U, U to $\phi$, is also an identity called the dual of the statement S.

**Remark :** The number of minsets generated by n sets is $2^n$.

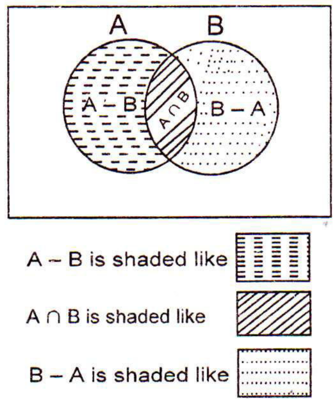
## 1.1.8 The Inclusion-Exclusion Principle

It is the most general form of addition principle for enumeration. As we know that number of elements of a finite set A is denoted by n (A) or $|A|$, so following results regarding number of elements should be kept in mind for doing problems :

1. $n(A \cup B) = n(A) + n(B) - n(A \cap B)$
2. $n(A \cup B) = n(A) + n(B) \Leftrightarrow$ A, B are disjoint sets.
3. $n(A \cup B) = n(A - B) + n(B - A) + n(A \cap B)$
4. $n(A) = n(A - B) + n(A \cap B)$
5. $n(B) = n(B - A) + n(A \cap B)$
6. $n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B)$
   $\qquad - n(B \cap C) - n(C \cap A) + n(A \cap B \cap C)$
7. $n(A' \cup B') = n(A \cap B') = n(U) - n(A \cap B)$
8. $n(A' \cap B') = n((A \cup B)') = n(U) - n(A \cup B)$
9. $n(A \cap B' \cap C') = n(A) - n(A \cap B) - n(A \cap C) + n(A \cap B \cap C)$

**Proof :** (1) We know that $A \cup B$ is the union of three disjoint sets

$A - B$, $A \cap B$ and $B - A$.




$\therefore \quad n(A \cup B) = n(A - B) + n(B - A) + n(A \cap B)$ (1)

Again A is union of $A - B$ and $A \cap B$, which are disjoint sets

$\therefore \quad n(A) = n(A - B) + n(A \cap B)$

Similarly, $n(B) = n(B - A) + n(A \cap B)$

Adding (2) and (3), we get

$$n(A) + n(B) = n(A - B) + n(B - A) + 2n(A \cap B)$$

$$= [n(A - B) + n(B - A) + n(A \cap B)] + n(A \cap B)$$

$\therefore \quad n(A) + n(B) = n(A \cup B) + n(A \cap B)$ $[\because$ of (1)$]$

$\Rightarrow \quad n(A \cup B) = n(A) + n(B) - n(A \cap B)$

Proof of (2) Since A and B are disjoint sets

$\therefore \quad A \cap B = \phi \Rightarrow n(A \cap B) = 0$

$\therefore \quad n(A \cup B) = n(A) + n(B) - 0$ or $n(A \cup B) = n(A) + n(B)$.

Proof of (6) L.H.S. $= n(A \cup B \cup C) = n[(A \cup (B \cup C)]$

$$= n(A) + n(B \cup C) - n[A \cap (B \cup C)]$$

$= n(A) + n(B) + n(C) - n(B \cap C) - n[(A \cap B) \cup (A \cap C)]$

$= n(A) + n(B) + n(C) - n(B \cap C) - [n(A \cap B) + n(A \cap C)$

$- n[(A \cap B) \cap (A \cap C)]$

$= n(A) + n(B) + n(C) - n(B \cap C) - n(A \cap B) - n(A \cap C) + n(A \cap B \cap C)$

$\therefore \quad n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C)$

$- n(C \cap A) + n(A \cap B \cap C)$.

**Example 1.10 :** In a group of 50 persons, 14 drink tea but not coffee and 30 drink tea. Find

(i) How many drink both tea and coffee ?

(ii) How many drink coffee but not tea ?

**Sol.** Let T denote the set of persons drinking tea and C denote the set of persons drinking coffee.

$\therefore$ $n(T \cup C) = 50,\ n(T) = 30,\ n(T \cap C^C) = 14$

(i) now $n(T \cap C^C) = n(T) - n(T \cap C)$

$\therefore$ $14 = 30 - n(T \cap C) \quad \Rightarrow n(T \cap C) = 16$

$\therefore$ number of persons drinking both tea and coffee = 16

(ii) Also $n(T \cup C) = n(T) + n(C) - n(T \cap C)$

$\therefore$ $50 = 30 + n(C) - 16 \quad \Rightarrow n(C) = 36$

$\therefore$ number of persons drinking coffee but not tea

$= n(C \cap T^C) = n(C) - n(C \cap T)$

$= n(C) - n(T \cap C) = 36 - 16 = 20$

**Example 1.11 :** A survey of 500 television watchers produced the following information:

285 watch football, 195 watch hockey, 115 watch basketball, 45 watch football and basketball, 70 watch football and hockey, 50 watch hockey and basketball, 50 do not watch any of the three games.

How many watch all the three games ? How many watch exactly one of the three games ?

**Sol.** Let F, H, B denote the sets of viewers who watch football, hockey, basketball respectively.

$\therefore$ $n(F) = 285,\ n(H) = 195,\ n(B) = 115,\ n(F \cap B) = 45,$

$n(F \cap H) = 70,\ n(H \cap B) = 50,\ n(F \cup H \cup B)^c = 50$

Also total number of viewers = 500

Now $n(F \cup H \cup B)^C = 50$

$\Rightarrow$ $500 - n(F \cup H \cup B) = 50$

$\Rightarrow$ $n(F \cup H \cup B) = 450$

$\Rightarrow$ $n(F) + n(H) + n(B) - n(F \cap H) - n(H \cap B) - n(B \cap F)$

$+ n(F \cap H \cap B) = 450$

$\Rightarrow$ $285 + 195 + 115 - 70 - 50 - 45 + n(F \cap H \cap B) = 450$

$\Rightarrow$ $n(F \cap H \cap B) = 20$

$\therefore$ number of viewers watching all the three games = 20.

Number of viewers watching football alone = $n(F \cap H^C \cap B^C)$

$= n(F) - n(F \cap H) - n(F \cap B) + n(F \cap H \cap B)$

$= 285 - 70 - 45 + 20 = 190$

Number of viewers watching hockey alone = $n(H \cap F^C \cap B^C)$

$= n(H) - n(H \cap F) - n(H \cap B + n(F \cap H \cap B)$

$= 195 - 70 - 50 + 20 = 95$

Number of viewers watching basketball alone $= n(B \cap H^C \cap F^C)$

$= n(B) - n(B \cap H) - n(B \cap F) + n(F \cap H \cap B)$

$= 115 - 50 - 45 + 20 = 40$

$\therefore$ number of viewers watching exactly one of the three games

$= 190 + 95 + 40 = 325.$

**Example 1.12 :** Find how many integers between 1 and 60 are neither divisible by 2 nor by 3 and nor by 5 ?

**Sol.** Let A, B and C be the set of integers between 1 and 60 divisible by 2, 3 and 5 respectively.

$$\therefore \quad n(A) = \left[\frac{60}{2}\right] = 30, n(B) = \left[\frac{60}{3}\right] = 20, n(C) = \left[\frac{60}{5}\right] = 12$$

$$\text{and} \quad n(A \cap B) = \left[\frac{60}{2 \times 3}\right] = 10, \; n(A \cap C) = \left[\frac{60}{2 \times 5}\right] = 6, n(B \cap C) = \left[\frac{60}{3 \times 5}\right] = 4$$

$$\therefore \quad n(A \cap B \cap C) = \left[\frac{60}{2 \times 3 \times 5}\right] = 2$$

Number of integers between 1 and 60 which are divisible by 2, 3 or 5 are

$= n(A \cup B \cup C)$

$= n(A) + n(B) + n(C) - n(A \cap B) + n(A \cap C) - n(B \cap C) + n(A \cap B \cap C)$

$= 30 + 20 + 12 - 10 - 6 - 4 + 2 = 44$

$\therefore$ the number of integers between 1 and 60 that are not divisible 2, 3 or 5

$= 60 - 40 = 16.$

## 1.1.9 Languages & Grammars

Let A be a set whose elements are called letters. Then, A is known as **Alphabet.**

A **word or string** on the set A is a finite sequence of its elements.

**For Example :** u = ababb, v = accbaaa are words on alphabet A = {a,b,c}

The empty sequence of letters is denoted by $\lambda$ and it is known as empty word.

The length of word is written as $|u|$ or $l(u)$.

If $u = a_1a_2...a_n$, then $l(u) = n$ and if $u = \lambda$, then $l(u) = 0$.

Now, we may define a language as –

**Language :** A language L over an alphabet A is collection of words on A. L is a subset of $A^{\alpha}$, where $A^{\alpha}$ is the set of all words on A.

**For Example :** If A = {a,b}, then $L_1$={a, ab, $ab^2$,...} which consists of all words beginning with a and followed by zero or more b's and $L_2$ = {$a^m b^n$; m > 0, n > 0] which consists of all words beginning with one or more a's followed by one or more b's, are languages over A.

**Grammar :** A grammar consists of four parts

(1) A finite set V of elements called variables or non-terminals.

(2) A finite set T of elements called Terminals.

(3) An element S in V called the star symbol.

(4) A finite set P of productions. A production is an ordered pair $(\alpha, \beta)$ usually written as $\alpha \to \beta$ where $\alpha, \beta$ are words in V, T. At least one of the $\alpha$ or $\beta$ must contain a variable. Such a Grammar is denoted by G (V, T, S, P)

Terminals are denoted by a, b, c, .......... .

Variables are denoted by A, B, C, ........... .

S denotes the start variable.

$\alpha, \beta$ will denote words in both variables and terminals.

We write, $\alpha \to (\beta_1, \beta_2, \ldots\ldots \beta_k)$ for

The production $\alpha \to \beta_1, \alpha \to \beta_2, \ldots\ldots\ldots, \alpha \to \beta_k$.

## Types of Grammars :

(1) A type O grammar has no restrictions on its productions.

(2) A grammar G is of Type-I if every production is of the form $\alpha \to \beta$ where $|\alpha| \leq |\beta|$.

(3) A grammar G is said to be of Type-2 if every production is of the form $A \to B$ i.e. the L.H.S is a non-terminal.

(4) A grammar G is said tobe of Type-3 if every production is of the form $A \to a$ or $A \to aB$ where L.H.S is a single non-terminal and R.H.S is a single terminal or terminal followed by a non-terminal or of the form $S \to \lambda$.

(5) A grammar G is said to be context sensitive if the production are of the form $\alpha A \alpha^1 \to \alpha\beta\alpha^1$ i.e. we replace the variable A by $\beta$ in a word only when A lies between $\alpha$ and $\alpha^1$.

(6) A grammar G is said to be context free if the productions are of the form $A \to \beta$. (i.e. Replace variable A by $\beta$ regardless of where A appears).

(7) A grammar G is said to be regular if the productions are of the form $A \to a$, $A \to aB$, $S \to \lambda$.

**1.1.10 Self Check Exercise**

1. If A, B are two sets, then show that $A \cup B = \phi \Leftrightarrow A = \phi, B = \phi$.

2. Is it true that power set of $A \cup B$ is equal to union of power sets of A and B ? Justify.

3. Prove the following :
   (i) $A \cap (A^C \cup B) = A \cap B$
   (ii) $A - (B \cap C) = (A-B) \cup (A-C)$
   (iii) $A - (B \cup C) = (A - B) \cap (A - C)$
   (iv) $A \cap (B - C) = (A \cap B) - (A \cap C)$
4. Let A = [+, –], and B = [00, 01, 10, 11]
   (a) List the elements of $A \times B$
   (b) How many elements do $A^4$ and $(A \times B)^3$ have ?
5. If A and B be non-empty subsets, then show that $A \times B = B \times A$ iff A = B.
6. Use PMI to show that, $n^3 + 2n$ is divisible by 3, where $n \in N$.
7. Prove that $n^2+n$ is even, where $n \in N$. (Use PMI)
8. A class has a strength of 70 students. Out of it 30 students have taken Mathematics and 20 have taken Mathematics but not Statistics. Find
   (a) The number of students who have taken Mathematics and Statistics?
   (b) How many of them have taken Statistics but not Mathematics ?
9. In a town of 10,000 families, it was found that 40% families buy newspaper A, 20% buy newspaper B and 10% buy newspaper C. 5% families buy A and B, 3% buy B and C, and 4% buy A and C. If 2% families buy all the newspapers, find the number of families which buy (i) A only (ii) B only (iii) none of A, B, and C.
10. Among integers 1 to 300, how many of them are divisible neither by 3, nor by 5, nor by 7 ? How many of them are divisible by 3 but not by 5, nor by 7 ?

**Suggested Readings :**

1. Dr. Babu Ram, Discrete Mathematics
2. C.L. Liu, Elements of Discrete Mathematics (Second Edition), McGraw Hill, International Edition, Computer Science Series, 1986.
3. Discrete Mathematics, S. Series.
4. Kenneth H. Rosen, Discrete Mathematics and its Applications, McGraw Hill Fifth Ed. 2003.

**LESSON NO. 1.2** **Author : Dr. Chanchal**

# THE BASICS OF COUNTING & PIGEONHOLE PRINCIPLE

**Structure :**



## 1.2.1 Introduction

Firstly, we define factorial n as :

**Def :** The product of all positive integers from 1 to n is called factorial n. It is denoted by $\underline{|n}$ or n !.

**For Example :** $\underline{|4} = 4.3.2.1 = 24$

**Note :** (i) $\underline{|n} = n(n-1)\,(n-2)\,(n-3).....3.2.1 = n\underline{|n-1}$

(ii) $\underline{|0} = 1, \underline{|1} = 1$

(iii) Factorial of proper fraction and negative integer is not defined.

**Example 1 :** Prove that $\frac{\underline{|2n}}{\underline{|n}} = 1.3.5........(2n-1)\,.\,2^n$

**Sol.** $$\text{L.H.S.} = \frac{\lfloor 2n}{\lfloor n} = \frac{1.2.3.4.5.6\ldots\ldots(2n-1)(2n)}{\lfloor n}$$

$$= \frac{[1.3.5\ldots\ldots\ldots(2n-1)][2.4.6\ldots\ldots\ldots(2n)]}{\lfloor n}$$

$$= \frac{[1.3.5\ldots\ldots(2n-1)]\ .\ 2^n[1.2.3\ldots\ldots\ldots n]}{\lfloor n}$$

$$= \frac{[1.3.5\ldots\ldots\ldots(2n-1)]\ .\ 2^n \lfloor n}{\lfloor n} = 1.3.5\ \ldots\ldots\ldots(2n-1)\ 2^n = \text{R.H.S.}$$

## 1.2.2 Fundamental Principle of Counting

If one operation can be performed in 'm' different ways and if corresponding to each of these m ways of performing the first operation, there are 'n' different ways of performing the second operation, then the number of different ways of performing the two operations taken together is $m \times n$.

**Example 2 :** How many numbers can be formed from the digits 1, 2, 3, 9 if repetition of digits is not allowed ?

**Sol. (a) Numbers with one digit :** There are four digits, hence four numbers of one digit can be formed with the help of these digits.

Hence, number of one digit numbers = 4.

**(b) Numbers with two digits :** First place of two digit number can be filled in 4 ways and the second place can be filled in 3 ways.

Hence, number of two digit numbers = 4 × 3 = 12.

**(c) Numbers with three digits :**

Number of three digits number = 4 × 3 × 2 = 24.

**(d) Number with four digits :**

Number of four digits numbers = 4 × 3 × 2 × 1 = 24.

Hence, total number of digits formed with the given digits

= 4 + 12 + 24 + 24 = 64.

**Example 3 :** Given 5 flags of different colours, how many different signals can be generated if each signal requires the use of 2 flags, one below the other ?

**Sol.** Number of flags = 5

Number of flags required for a signal = 2

First place of signal can be filled in 5 ways and corresponding to each way of filling the first place, there are four ways of filling the second place.

$\therefore$ by fundamental principle of counting.

total number of signals generated = 5 × 4 = 20.

### 1.2.3 Permutation

**Def :** It is an arrangement that can be made by taking some or all of a number of given things. It is denoted by ${}^nP_r$ which means number of permutations of n different things taken 'r' at a time. Further, ${}^nP_r = \dfrac{\lfloor n}{\lfloor n-r}$.

**Illustration :** Consider three letters a, b, c. Now, the permutations of three letters taken two at a time are : ab, bc, ca, ba, cb, ac, which are 6 in number.

Mathematically, ${}^nP_r = {}^3P_2 \dfrac{\lfloor 3}{\lfloor 3-2} = \dfrac{\lfloor 3}{\lfloor 1} = 6$

**Note :** ${}^nP_r$ is also writen as P (n, r)

**Example 4 :** Find n if P (2n, 3) = 100 P (n, 2).

**Sol.** Since P (2n, 3) = 100 P (n, 2)

$\therefore$ ${}^{2n}P_3 = 100\ {}^nP_2 \Rightarrow (2n)(2n-1)(2n-2) = 100\, n\, (n-1)$

$\Rightarrow$ $4n(n-1)(2n-1) = 100n(n-1) \Rightarrow 2n-1 = 25$

$\Rightarrow$ $n = 13$

### 1.2.4 Practical Problems Involving Permutation

**Example 5 :** How many words, with or without meaning, can be formed using all the letters of the word EQUATION, using each letter exactly once. Further, how many words can be formed if each word is to start with a vowel.

**Sol.** Number of letters is EQUATION = 8

No. of letters to be taken at a time = 8

$\therefore$ required no. of words $= {}^8P_8 = \dfrac{\lfloor 8}{\lfloor 0} = 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 40320$

Further, vowels in EQUATION are E,U, A, I, O i.e. 5 vowels. If the first place is to be filled with vowel, it can be filled in 5 ways. Now, remaining 7 places can be filled up with 7 letters in $\lfloor 7$ or ${}^7P_7$ ways.

$\therefore$ required no. of words $= 5 \times \lfloor 7 = 5 \times (7\times6\times5\times4\times3\times2\times1) = 25200$

**Example 6 :** In how many ways can 5 books on Chemistry and 4 books on Physics be arranged on a shelf so that the books on same subject remain together ?

**Sol.** Consider the 5 books on Chemistry as one book and also 4 books on Physics as one book.

$\therefore$ two books can be arranged in $\underline{|2}$ ways

Also the 5 books on Chemistry can be arranged among themselves in $\underline{|5}$ ways and 4 book on Physics in themselves in $\underline{|4}$ ways.

$\therefore$ required number of ways = $\underline{|2} \times \underline{|5} \times \underline{|4}$

$= (2 \times 1) \times (5 \times 4 \times 3 \times 2 \times 1) \times (4 \times 3 \times 2 \times 1)$

$= 2 \times 120 \times 24 = 5760.$

**Example 7 :** In how many ways can 4 boys and 3 girls be seated in a row so that two girls are together ?

**Sol.** Let 4 boys the $B_1$, $B_2$, $B_3$, $B_4$

$\times\ B_1 \times B_2 \times B_3 \times B_4 \times$

$\because$ no two girls are together

$\therefore$ three girls can be arranged in 5 '×' marked places in ${}^5P_3$ ways.

Also 4 boys can be arranged among themselves $\underline{|4}$ ways

$\therefore$ required number of ways = ${}^5P_3 \times \underline{|4}$

$= (5 \times 4 \times 3) \times (4 \times 3 \times 2 \times 1)$

$= 60 \times 24 = 1440$

**Example 8 :** How many numbers greater than 40000 can be formed using the digits 1, 2, 3, 4 and 5 if each digit is used only once in each number ?

**Sol.** Given digits are 1, 2, 3, 4, 5

$\therefore$ number of given digits = 5

Number of digits to be taken at a time = 5

Since number is to be greater than 4000

$\therefore$ first digit from left should be either 4 or 5

i.e. first place can be filled in 2 ways.

Remaining 4 places with 4 digits can be filled in ${}^4P_4$ ways

$\therefore$ required numbers = $2 \times {}^4P_4 = 2 \times (4 \times 3 \times 2 \times 1) = 2 \times 24 = 48.$

**Example 9 :** How many different signals can be formed with five given flags of different colours ?

**Sol.** Number of flags = 5

A signal may formed by hoisting any number of flags at a time.

Number of signals by hoisting one flag at a time = ${}^5P_1$

Number of signals by hoisting two flags at a time = ${}^5P_2$

Number of signals by hoisting three flags at a time = ${}^5P_3$

Number of signals by hoisting four flags at a time = ${}^5P_4$

Number of signals by hoisting five flags at a time = ${}^5P_5$

$\therefore$ total number of signals formed

$= {}^5P_1 + {}^5P_2 + {}^5P_3 + {}^5P_4 + {}^5P_5$

$= 5 + 20 + 60 + 120 + 120 = 325.$

**Result :** The number of permutations of n things taken all at a time when p of them are alike and of one kind, q of them are alike and of second kind, all other being different is given by $\dfrac{\lfloor n}{\lfloor p \times \lfloor q}$.

**Example 10 :** How many permutations of the letter of word APPLE are there ?

**Sol.** No. of given letters = 5

No. of P's = 2

$$\therefore \text{ required no. of permutations } = \frac{\lfloor 5}{\lfloor 2} = \frac{5 \times 4 \times 3 \times 2 \times 1}{2 \times 1} = 60$$

**Example 11 :** How many numbers greater than 1000000 can be formed by using the digits 1, 2, 0, 2, 4, 2, 4.

**Sol.** Given digits are 1, 2,0, 2, 4, 2, 4

$\therefore$ total number of digits = 7

with Number of 2's = 3 and Number of 4's = 2

Number of digits to be taken at a time = 7

$$\therefore \quad \text{numbers formed } = \frac{\lfloor 7}{\lfloor 3 \times \lfloor 2} = \frac{7 \times 6 \times 5 \times 4 \times \lfloor 3}{\lfloor 3 \times (1 \times 2)} = 420$$

These numbers also include those numbers which have 0 at the extreme left position.

$$\text{Numbers having 0 at the extreme left position } = \frac{\lfloor 6}{\lfloor 3 \times \lfloor 2} = \frac{6 \times 5 \times 4 \times \lfloor 3}{\lfloor 3 \times (1 \times 2)} = 60$$

$\therefore$ required number of numbers = 420 – 60 = 360.

### 1.2.4.1 Circular Permutations

Find the number of ways in which n persons can be arranged at a round table.

**Proof :** When n persons are sitting around a circular table, then there is no first and last person. Let us fix the position of one person. The remaining (n–1) persons can be arranged in the remaining (n–1) places in ${}^{n-1}P_{n-1}$ i.e., $\underline{|n-1}$ ways.

$\therefore$ required number of ways = $\underline{|n-1}$.

## Clockwise and Anti-clockwise Permutations

The total number of circular permutations can be divided into two types :

(i) Clockwise (ii) Anti-clockwise.

In two such arrangements each person has the same neighbour though in the reverse order and either of these arrangements can be obtained from the other by just over-turning the circle. If in this case, no distinction is made between clockwise and anti-clockwise arrangements then the two such arrangements are considered as only one distinct arrangement.

Hence the number of circular permutations in such cases $= \frac{1}{2}\underline{|n-1}$

**Note.** Questions on necklaces with beads if different colours are to be tackled, by the above formula, as in this case also there is no difference between clockwise and anticlockwise arrangements.

**Example 12 :** In how many ways can 8 girls be seated at a round table provided Parveen and Vipul are not to sit together ?

**Sol.** Total number of girls = 8.

$\therefore$ number of ways in which they can be arranged on a round table

$$= \underline{|8-1} = \underline{|7} = 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 5040$$

Consider two girls Parveen and Vipul as one. Therefore 7girls can be arranged in $\underline{|7-1} = \underline{|6}$ ways. Also the two girls can be arranged among themselves in $\underline{|2}$ ways.

$\therefore$ number of arrangements in which two particular girls are always together

$$= \underline{|6} \times \underline{|2} = (6 \times 5 \times 4 \times 3 \times 2 \times 1) \times (2 \times 1) = 1440$$

$\therefore$ required number of arrangements = 5040 – 1440 = 3600.

## 1.2.5 Combination

**Def :** It is a group (or selection) that can be made by taking some or all of a number of given things at a time. It is denoted by ${}^{n}C_{r}$ which means number of combinations of n different things taken 'r' at a time. Further, ${}^{n}C_{r} = \frac{\underline{|n}}{\underline{|r}\,\underline{|n-r}}$.

**Illustration :** Consider three letters a,b, c. The groups of there 3 letters taken two at a time are ab, bc, ca. As far as group is concerned ac or ca is the same group because in a group, we are concerned with the number of things contained unlike with the case of arrangement where we have to consider the order of things also.

**Note :** (i) $^nC_r = {}^nC_{n-r}$

(ii) $^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$

(iii) The total number of combinations of n different things by taking some or all at a time i.e. $^nC_1 + {}^nC_2 + {}^nC_3 + \dots\dots\dots + {}^nC_n$ is given by $2^n-1$.

The above results can be proved very easily and left as an exercise for the reader.

## 1.2.6 Practical Problems Involving Combinations

**Example 13 :** A mathematics paper consists of 10 questions divided into two parts I and II. Each part containing 5 questions. A student is required to attempt 6 questions in all, taking at least 2 questions from each part. In how many ways can the student select the questions ?

**Sol.** Number of questions in part I = 5

Number of question in part II = 5

| **Part I** | **Part II** | **Total** |
|---|---|---|
| 2 | 4 | 6 |
| 3 | 3 | 6 |
| 4 | 2 | 6 |

Number of questions to be attempted = 6

$\because$ each selection contains at least 2 from each part

$\therefore$ different possibilities are

(i) 2 from part 1, 4 from part II

(ii) 3 from part 1, 3 from part II

(iii) 4 from part 1, 2 part II

$\therefore$ required number of ways

$$= {}^5C_2 \times {}^5C_4 + {}^5C_3 \times {}^5C_3 + {}^5C_4 \times {}^5C_2$$

$$= {}^5C_2 \times {}^5C_1 + {}^5C_2 \times {}^5C_2 + {}^5C_1 \times {}^5C_2$$

$$= \frac{5\times4}{1\times2} \times \frac{5}{1} + \frac{5\times4}{1\times2} \times \frac{5\times4}{1\times2} + \frac{5}{1} \times \frac{5\times4}{1\times2}$$

$$= 10\times5 + 10 \times 10 + 5 \times 10 = 50 + 100 + 50 = 200.$$

**Example 14 :** A committee of 5 is to be selected from among 6 boys and 5 girls. Determine the number of ways of selecting the committee if it is to consist of at least 1 boy and 1 girl.

**Sol.** Number of boys = 6, Number of girls = 5

| Boys=6 | Girls=5 |
|---|---|
| 1 | 4 |
| 2 | 3 |
| 3 | 2 |
| 4 | 1 |

Committee is to be formed of 5.

$\because$ committee is to include at least 1 boy and 1 girl

$\therefore$ different possibilities are

(i) 1 boys, 4 girls (ii) 2 boys, 3 girls

(iii) 3 boys, 2 girls (iv) 4 boys, 1 girl

$\therefore$ required number of ways

$= {}^6C_1 \times {}^5C_4 + {}^6C_2 \times {}^5C_3 + {}^6C_3 \times {}^5C_2 + {}^6C_4 \times {}^5C_1$

$= {}^6C_1 \times {}^5C_1 + {}^6C_2 \times {}^5C_2 + {}^6C_3 \times {}^5C_2 + {}^6C_2 \times {}^5C_1$

$$= \frac{6}{1} \times \frac{5}{1} + \frac{6\times5}{1\times2} \times \frac{5\times4}{1\times2} + \frac{6\times5\times4}{1\times2\times3} \times \frac{5\times4}{1\times2} + \frac{6\times5}{1\times2} \times \frac{5}{1}$$

$= 30 + 150 + 200 + 75 = 455.$

**Example 15 :** The number of diagonals of a polygon is 20. Find the number of its sides.

**Sol.** Let number of sides of polygon = n, $\therefore$ Number of points = n

Number of lines formed $= {}^nC_2 = \dfrac{n\,(n-1)}{2}$

$\therefore$ number of diagonals $= \dfrac{n\,(n-1)}{2} - n$

From given condition, $\dfrac{n\,(n-1)}{2} - n = 20$

$\therefore$ $n^2 - n - 2n = 40$ $\Rightarrow$ $n^2 - 3n - 40 = 0$

$\Rightarrow$ $(n-8)\,(n + 5) = 0$ $\Rightarrow$ $n = 8, -5$

Rejecting n = –5 as number of sides cannot be negative, we get, n = 8

$\therefore$ number of sides = 8.

**Example 16 :** Ram has 5 friends. In how many ways can be invite one or one of them to a party ?

**Sol.** Number of friends = 5

Ram can invite one friend, two friends, three friends, four friends, four friends or five friends.

$\therefore$ required number of ways $= {}^5C_1 + {}^5C_2 + {}^5C_3 + {}^5C_4 + {}^5C_5$

$= {}^5C_1 + {}^5C_2 + {}^5C_2 + {}^5C_1 + {}^5C_0$

$$= \frac{5}{1} + \frac{5\times4}{1\times2} + \frac{5\times4}{1\times2} + \frac{5}{1} + 1$$

$= 5 + 10 + 10 + 5 + 1 = 31.$

## 1.2.7 Theory of Probability

The word *"proable"* means "likell", or "most likely to be true". In other words, we have an uncertain situation, where we cannot predict in advance what is going to happen. The probability theory provides a mathematical model to study the uncertain situations. Consider the following simple examples. Suppose, we toss a coin on the floor. Then, either a head or a tail may show up. We cannot predict in advance whether a head or a tail will show up. Consider now a die. A die is a cube which has six faces. Let these faces be numbered as 1,2,3,4, 5 and 6. If the die is thrown, then any one of the six faces can turn up. Again, we cannot predict in advance which number is going to turn up. Consider a pack of 52 playing cards. Assume that we have shuffled the pack of cards and a card is drawn. Again, we are not sure which card it will be.

Now, we define certain terms related to probability.

**Random Experiment :**

*Random* means "haphazard". Any experiment happening under uncertain situations or conditions is called a *random trial* or a *random experiment.* It is also known as called an *experiment of chance.* The three examples which we have described earlier viz. - result of tossing a coin, result of throwing a dice, result of drawing a card from a pack of 52 playing cards - are random events or random experiments. In all these experiments, there are more than one possible outcomes, but we are not sure which one of these outcomes will actually occur. Thus, the theory of probability may be defined as that branch of mathematics where we investigate and discuss various rules for random experiements.

**Event :**

Any question that we ask with regard to a random experiment defines an event.

**Elementary Events :**

Suppose that we have conducted a random experiment. It is completely defined when we know all the possible outcomes. Each outcome of this experiment is called an elementary event. Suppose that a dice is thrown. The appearance of any number $i$ ($i$ = 1,2,34,5,6) is an elementary event and there are six elementary events in this experiment. If a coin is tossed, then either a head ($H$) or a tail ($T$) may turn up. Therefore, head or tail are the two elementary events of the experiment of tossing a coin.

**Sample Space :**

Sample space of a random experiment is the set of all the possible outcomes, that is, the set of all elementary events of that experiment. We assume that this set is finite. Let *E* denote the random experiment and $e_1$, $e_2$,.....$e_m$ denote all the possible outcomes. Then, the simple space *S* of the experiment is the set $S$ = ($e_1$, $e_2$,.....$e_m$). Obviously, each element of the set is a possible outcome and each outcome of a trial corresponds to only one element of the set *S*.

Consider the following random experiments :

1. A coin is tossed. There are two possible outcomes, either a head (*H*) or a tail (*T*) may turn up. Hence, the sample space of the experiment contains two elementary events *H, T* and therefore, *S* = (*H,T*).
2. Consider the random experiment of throwing a dice and noting the resulting number. The experiment can result in turning up of any one of the six numbers 1,2,3,4,5,6, which are the elementary events of the experiment. Hence, the sample space of the experiment is *S* = {1,2,3,45,6}.
3. Let two coins be thrown simultaneously. Then, the first coin may show up either *H* or *T* and the second coin may also show up either *H* or *T*. Denote by HT, the case of a head turning up on the first coin and a tail turning up on the second coin. Similarly, we define *HH*, *TH* and *TT*. The possible outcomes of the experiment are *HH*, *HT*, *TH* and *TT*, which are the elementary events of the experiment. Hence, the sample space is *S* = {*HH*, *HT*, *TH*, *TT*}.

**Sure Event (Universal event) :**

The sample space S of an experiment is the set of all possible outcomes. Since, a set is also a subset of itself, *S* can also be considered as representing an event associated with the experiment. Now, since every outcome belongs to *S*, the event represented by the set *S* always occurs. Therefore, the event represented by *S* is called a sure event.

**Impossible event :**

An empty set $\phi$ is always a subset of a set *S*. Hence, the empty set $\phi$ can always be considered as representing an event of an experiment. But, there is no outcome of the experiment which can belong to $\phi$. Hence, the event represented by $\phi$ is called an impossible event.

Consider again, the example of throwing two dice simultaeously. Let an event *B* be defined as "the sum of the numbers on the faces is greater than or equal to 2". Since, the sum of the smallest numbers of the two faces is 2, the sum of the two numbers appearing on the two faces is always greater than or equal to 2. Hence, the set of outcomes is same as the sample space *S*. Therefore, the event *B* is a sure event. Define another event *A* as "the sum of the numbers is greater than 12". Since, the sum of the largest numbers of the two faces is 12, the sum of the two numbers appearing on the two faces can never be greater than 12, Hence, the set of outcomes of the event $A$ is $\phi$. Therefore, $A$ is an impossible event.

**Equally likely events**

Let $S$ be the sample space of a random experiment. If all the elementary events of $S$ have the same chance of occuring, then the events are said to be equally likely events.

**Mutually exclusive events**

Consider the example of throwing two dice simultaneously, and the sum of the two numbers is noted. Define the events $A$ and $B$ as

$A$ : the sum of the two numbers appearing on the dice is less than or equal to 4.

$B$ : the sum of the two numbers appearing on the dice is greater than 9.

The set $E_1$ of outcomes of the event $A$ is $E_1 = \{2, 3, 4\}$. The set $E_2$ of outcomes of the event $B$ is $E_2 = \{10, 11, 12\}$. The sample space of the experiment is

$$S = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}.$$

We find that $E_1 \cap E_2 = \phi$ and hence, the sets $E_1, E_2$ are disjoint. We then say that the event $A$ and $B$ are mutually exclusive. That is, when one event has occured, the other event cannot occur or the two events cannot occur together. If the events associated with an experiment are mutually exclusive, then the subsets of the sample space representing the events are disjoint. The converse is also true. That is, if the subsets representing the events of an experiment are disjoint, then the events are said to be mutually exclusive.

**Mutually exhaustive events (Definition) :**

Let $S$ be the sample space of a random experiment and $A_1, A_2, \ldots\ldots A_m$ be the events defined on the sample space. If $A_1 \cup A_2 \cup \ldots \cup A_m = S$, then the events are said to be exhaustive. If further $A_1 \cap A_j = \phi$, $i \neq j$, then the events are said to be mutually exclusive and exhaustive.

**Combination of events**

We now consider combination of events defined in an experiment. This can be done by using the operations "or", "and", "not". Let us first consider an example. Consider the experiment of throwing two dice simultaneously and noting the total of the numbers that have turned up. Define the events

$A$ : the sum of the two numbers is less than or equal to 5,

$B$ : the sum of the numbers satisfy, $4 \leq \text{sum} \leq 8$,

$C$ : the sum of the numbers satisfy, $5 < \text{sum} < 8$.

Let $E_1$, $E_2$ and $E_3$ denote the sets of outcomes of the events $A$, $B$ and $C$ respectively. Now, $E_1 = \{2,3,4,5\}$, $E_2 = \{4,5,6,7,8\}$, $E_3 = \{6,7\}$, while the sample space is $S = \{2,3,\ldots.12\}$. We define the event $A$ or $B$ as the event which occurs when either $A$ or $B$ or both occur. In set notation, we denote this operation as $A \cup B$. The set representing $A \cup B$ in the above example is $E = E_1 \cup E_2 = \{2,3,4,5,6,7,8\}$.

We define a new event *A* and *B*, as the event which occurs when *A* and *B* both occur. In set notation, we denote this operation by $A \cap B$. In the above example, the set representing $A \cap B$ is $E = E_1 \cap E_2 = \{4,5\}$.

We define the event *not A*, as the event which occurs when *A* does not occur. If $E_1$ is the set representing the event *A*, then the set representing *not A* contains all elements of the sample space *S* which do not belong to $E_1$. Thus, *not A* is represented by the *complement* in *S* of the set $E_1$, which can be written as $E_1^c$ or $\vec{E}_1$. Often, *not A* is also called the *complementary event* of *A*, or the *negation of A*.

Suppose that $E_2$, $E_3$ are the subsets of a sample space and we have $E_3 \subset E_2$. The subsets $E_2$ and $E_3$ represent the outcomes of the events B and C respectively. We define the event $C \subset B$ as "the event C implies the event B" or if the event C occurs, then the event B must occur.

We also define an event A - B as "the event A but not B".

**Probability of an Event :**

To every event in a random experiment we attach a numerical value which is called its probability. We denote the probabilities of the events *A* and *B* by *P(A)* and *P(B)* respectively. We say that $P(A) > P(B)$, if the event *A* is more likely to occur, than the event *B*. Obviously, every event is more likely to occur than the impossible event ad is less likely to occur than the sure event. Hence, the impossible event must have the smallest probability and the sure event must have the largest probability. By convention, we assign the value 0 to the probability of the impossible event and the value 1 to the probability of the sure event. Hence, the probability of an event *A* satisfies the inequality.

$$0 \leq P(A) \leq 1.$$

This result is an *axiom* of calculus of probability.

We now need a rule to compute the value of the probability of an event.

Let the sample space *S* contain *n* elementary events $e_i$, that is

$$S = \{e_1, e_2, e_3, \ldots\ldots e_n\}.$$

By definition, $P(e_i) \geq 0$, i = 1,2,......,n. Assume now, that all the elementary events are equally likely to occur when the experiment is performed.

Let the set of outcomes *E* represent an event *A*. If the experiment produces an outcome which belongs to *E*, then the event *A* is said to have occured. Let *n* and *m*, ($m \leq n$) be the number of elementary events in the sample space *S* and *E* respectively. Since, all the elementary events are equally likely to occur, we define the probability of the event A as

$$P(A) = p = \frac{number\ of\ events\ favourable\ to\ A}{total\ number\ of\ equally\ likely\ events} = \frac{m}{n}$$

**Example 1.** What is the chance that a leap year selected at random will contain 53 Sundays. (B.C.A. 2011)

**Sol.** Leap year contains 366 days.

$\therefore$ there are 52 complete weeks and two days other. The following are the possibilities for these two 'over' days :

(i) Sunday and Monday (ii) Monday and Tuesday
(iii) Tuesday and Wednesday (iv) Wednesday and Thursday
(v) Thursday and Friday (vi) Friday and Saturday
(vii) Saturday and Sunday.

Now there will be 53 Sundays in a leap year when one of the two over days is a Sunday.

$\therefore$ out of 7 possibilities, two are favourable to this event.

$\therefore$ required probability = $\frac{2}{7}$.

**Example 2.** Two cards are drawn at random from a well-shuffled pack of 52 cards. Show that the chances of drawing two aces is $\frac{1}{221}$.

**Sol.** Total number of cards = 52.

Two cards out of 52 cards can be drawn in ${}^{52}C_2$ ways.

$\therefore$ total number of outcomes = ${}^{52}C_2 = \frac{52 \times 51}{1 \times 2} = 1326$.

Now 2 aces out of 4 aces can be drawn in ${}^{4}C_2$ ways.

$\therefore$ total number of favourable outcomes = ${}^{4}C_2 = \frac{4 \times 3}{1 \times 2} = 6$.

$\therefore$ required probability = $\frac{6}{1326} = \frac{1}{221}$ .

**Art-1. Addition Theorem for Mutually Exclusive Events**

If A and B are two mutually exclusive events associated with a random experiment, then

$$P(A \text{ or } B) = P(A) + P(B)$$

**Proof :** Let $n$ be the total number of exhaustive, equally likely cases of the experiment. Let $m_1$ ad $m_2$ be the number of cases favourable to the happening of the events A and B respectively.

$\therefore$ $P(A) = \frac{m_1}{n}$, $P(B) = \frac{m_2}{n}$.

Since the events A and B mutually exclusive.

$\therefore$ there cannot be any sample point common to both events A and B.

$\therefore$ the event A or B can happen in exactly $m_1 + m_2$ ways.

$\therefore$ $P(A \text{ or } B) = \frac{m_1 + m_2}{n} = \frac{m_1}{n} + \frac{m_2}{n} = P(A) + P(B)$

$\therefore$ P(A or B) = P(A) + P(B)

**Note :** The result of this theorem can be extended to any number of mutually exclusive events.

**Art-2.** If A and B are two events associated with a random experiment, then

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

**Proof :** Let $n$ be the total number of exhaustive, equally likely cases of the experiment.

Let $m_1$ and $m_2$ be the number of cases favourable to the happening of the events A and B respectively.

$\therefore$ $P(A) = \frac{m_1}{n}$, $P(B) = \frac{m_2}{n}$.

Let $m_3$ be the number of sample points common to both A and B.

$\therefore$ $P(A \cap B) = \frac{m_3}{n}$.

Now $m_3$ sample points, which are common to both the events A and B, are included in the events A and B separately.

$\therefore$ total number of sample points in the event $A \cup B = m_1 + m_2 - m_3$

$$\therefore \quad P(A \cup B) = \frac{m_1 + m_2 - m_3}{n} = \frac{m_1}{n} + \frac{m_2}{n} - \frac{m_3}{n} = P(A) + P(B) - P(A \cap B)$$

$$\therefore \quad P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

**Cor.** If A and B are mutually exclusive events,

then $A \cap B = \phi$

$\therefore$ $P(A \cap B) = 0$

$\therefore$ $P(A \cup B) = P(A) + P(B)$

Now, the students can easily prove that :

(a) $P(\bar{A}) = 1 - P(A)$

(b) $P(\bar{A} \cap B) = P(B) - P(A \cap B)$

(c) $P(A \cap \bar{B}) = P(A) - P(A \cap B)$

(d) $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$

**Example 3.** Find the probability of 4 turning for at least once in two tosses of a fair die.

**Sol.** Here S = {(1,1), (1,2),.....(6,5), (6,6)}

Let two events A and B be

A : 4 on first die

B : 4 on second die

$\therefore$ A = {(4,1), (4,2), (4,3), (4,4), (4,5), (4,6)}

B = {(1,4), (2,4), (3,4), (4,4), (5,4), (6,4)}

$A \cap B$ = {(4,4)}

$\therefore$ $P(A) = \frac{6}{36}$, $P(B) = \frac{6}{36}$, $P(A \cap B) = \frac{1}{36}$

P (4 at least on one die) = $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$$= \frac{6}{36} + \frac{6}{36} - \frac{1}{36} = \frac{11}{36}.$$

**Independent Events**

Two events E and F defined on the sample space S of a random experiment are said to be independent if

$$P(E \text{ and } F) = P(E)\,P(F)$$

Two events E and F are said to be dependent if they are not independent

i.e. if $P(E \cap F) \neq P(E)\,P(F)$

**Example 4.** A die is thrown and 6 possible outcomes are assumed to be equally likely. If E is the event : "the number appearing is a multiple of 3" and F the event : "the number appearing is even". Show that E and F are independent events.

**Sol.** $P(E) = P(\{3,6\}) = \frac{2}{6} = \frac{1}{3}$

$P(F) = P(\{2,4,6\}) = \frac{3}{6} = \frac{1}{2}$

$P(E \cap F) = P(\{6\}) = \frac{1}{6}$

Now $\frac{1}{6} = \frac{1}{3} x \frac{1}{2}$

$\therefore P(E \cap F) = P(E).\ P(F)$

**Multiplication Theorem on Probability**

If E and F are two events associated with a random experiment, then

$P(E \cap F) = P(E)\,P(F|E)$, provided $P(E) \neq 0$.

or $P(E \cap F) = P(F)\,P(E|F)$, provided $P(F) \neq 0$.

**Example 5.** A bag contains 5 white, 7 red and 4 black balls. If four balls are drawn one by one with replacement, what is the probability that none is white?

**Sol.** Number of white balls = 5

Number of black balls = 7

$\therefore$ total number of balls = 5+7+4=16

P(all four balls as none-white) = $\frac{11}{16} x \frac{11}{16} x \frac{11}{16} x \frac{11}{16} = \left(\frac{11}{16}\right) 4$

**Example 6.** A husband and wife appear in an interview for two vacancies in the same post. The probability of husband's selection is $\frac{1}{7}$ and that of wife's selection is $\frac{1}{5}$. What is the probability that

(i) both of them will be selected? (ii) only one of them will be selected?

(iii) none of them will be selected?

## 1.2.8 Pigeonhole Principle

**Simple Form :** If n pigeons are assigned to m pigeonholes and m < n, then there is at least one pigeonhole that contains two or more pigeons.

**Proof :** Label n pigeons with the numbers 1 to n and m pigeonholes with the numbers 1 to m. Starting with pigeon 1 and pigeonhole 1, assign each pigeon in order to the pigeonhole with the same number. So we can assign as many pigeons as possible to distinct pigeonholes. Since the number m of pigeonholes is less than the number n of the pigeons, so n – m pigeons are left that are not assigned to a pigeonhole. Therefore, there is atleast one pigeonhole that will be assigned one or more than one pigeon again.

$\therefore$ there is at least one pigeonhole that contains two or more pigeons.

**Extended Form :** If n pigeons are assigned m pigeonholes, where n is sufficiently large as compared to m, then one of the pigeonholes must contain at least $\left[\frac{n-1}{m}\right]+1$ pigeons.

**Proof :** Assume that the result is false

$\therefore$ each pigeonhole does not contain more than $\left[\frac{n-1}{m}\right]$ pigeon.

$\therefore$ maximum possible number of pigeons $=\left[\frac{n-1}{m}\right]m \leq \frac{n-1}{m}\,.\,m$

$= n - 1$

This contradicts the given result that number of pigeons is n.

$\therefore$ our supposition is wrong.

Hence the result.

**Example 7 :** Use Pigeonhole Principle to show that if seven numbers from 1 to 12 are chosen, then two of them will add upto 13.

**Sol.** The sets which add upto 13 are

$\{1, 12\}, \{2, 11\}, \{3, 10\}, \{4, 9\}, \{5, 8\}, \{6, 7\}$.

By Pigeonhole principle, if we have to choose seven numbers then we must take at least two numbers belonging to one set. Thus two of the seven numbers will definitely add upto 13.

**Example 8 :** Use Pigeonhole Prinicple to prove that an injective mapping cannot exist between a finite set A and a finite set B if Cardinality of A is greater than Cardinality of B.

**Sol.** Let n (A) = a, n(B) = b where a > b.

Consider elements of Set B as pigeonholes and elements of set A as pigeons. As no. of pigeons are more than pigeon holes so at least two pigeons will have same pigeonholes or we can say $\exists$ x, y $\varepsilon$ A such that f (x) = f(y), but x $\neq$ y.

$\therefore$ f : A $\rightarrow$ B is not injective.

**Example 9 :** How many people among 200000 people are born at same time (hour, minute, seconds) ? Use Pigeonhole principle to find it.

**Sol.** Total number of persons = 200000

Total number of seconds in a day = 24 × 60 × 60 = 86,400

Here, we have to assign a time to each person

So person are like pigeons, time is like pigeonhole.

Number of pigeons (n) = 200000

Number of pigenholes (m) = 86,400

Min. number of persons having same birthday

$$= \left[\frac{n-1}{m}\right] + 1 = \left[\frac{200000-1}{86400}\right] + 1 = \left[\frac{1,99,999}{86400}\right] + 1 = 2 + 1 = 3 .$$

## 1.2.9 Self Check Exercise

1. Prove that $C(2n, n) = \dfrac{2^n\,[1.3.5..........(2n-1)]}{\underline{|n}}$
2. In how many different ways, the letters of the word ALGEBRA can be arranged in a row if two A's are never together ?
3. Find the number of different 8 letter words formed from the letters of word TRIANGLE if each word is to have both consonants and vowels together.
4. In how many ways 4 boys and 4 girls be seated at a round table provided each boy is to be between two girls ?
5. A group consists of 4 girls and 7 boys. In how many ways can a team of 5 numbers be selected if the team has (i) no girls ? (ii) atleast one boy and one girl ?
6. There are 15 points in a plane 1 no three of which are in the same straight line excepting 4, which are collinear. Find the no. of (i) straight lines (ii) triangles, formed by joining them.
7. A sport team of 11 students is to be constituted, choosing atleast 5 from class XI and atleast 5 from class XII. If there are 20 students in each of these classes, in how many ways can the team be constituted.
8. How many people must you have to guarantee that atleast 12 of them will have birthdays on the same day of the week ? Use pigeonhole principle.
9. Use pigeonhole principle to show that if seven numbers from 1 to 12 are chosen, then two of them will add upto 13.

**Suggested Readings :**


1. Dr. Babu Ram, Discrete Mathematics
2. C.L. Liu, Elements of Discrete Mathematics (Second Edition), McGraw Hill, International Edition, Computer Science Series, 1986.
3. Discrete Mathematics, S. Series.
4. Kenneth H. Rosen, Discrete Mathematics and its Applications, McGraw Hill Fifth Ed. 2003.

**LESSON NO. 1.3** **Author : Dr. Chanchal**

# RELATIONS AND FUNCTIONS

**Structure :**



## 1.3.1 Introduction

As we have already studied about the cartesian product of two sets in set theory, discussed in Lesson No. 1. In continuation to that, we can define a relation as :

**Def. Relation :** A relation from a set A to a set B is defined as a subset of A × B. Therefore each subset of A × B is a relation from A to B. If R is a relation from a set A to set B and if $(a, b) \in R$ for some $a \in A$ and $b \in B$, then we say that a is related to b and we write it as a R b. If $(a, b) \notin R$ then we say that a is not related to b and we write it as $a \not{R}\ b$.

### Domain and Range of a Relation

If R is a relation from a set A to a set B. Then the set of the first components of the elements of R is called the domain of R and the set of the second components of the elements of R is called the range of R.

Thus, domain of R = { a : (a, b) $\in$ R}, and range of R = {b : (a, b) $\in$ R}.

If R is a relation from a set A to the set A, then R is called a relation on A. Thus a relation on a set A is defined as any subset of A × A.

**For Example :** For any a, b $\in$ N, the set of natural numbers, define a relation R by a R b if a divides b.

Then, R = { (1, 1), (1, 2), (1, 3), ……, (2, 2), (2, 4),…….(3, 3), (3, 6),……..}

R is clearly a subset of N × N and hence a relation on N.

Here, (1, 2) $\in$ R since 1 divides 2 but (2, 1) $\notin$ R since 2 does not divide 1.

### 1.3.2 Types of Relation

### I. Reflexive Relation

**Def. :** A relation R on a set A is called a reflexive relation if (x, x) $\in$ R for all x $\in$ A i.e., if x R x for every x $\in$ A.

**For Example :** Let A = {1, 2}.

Then A × A = {(1, 1), (1, 2), (2, 1), (2, 2)}.

Let R = {(1, 1), (2, 2), (1, 2)}.

Clearly R $\subseteq$ A × A and so R is a relation on the set A.

Since (x, x) $\in$ R $\forall$ x $\in$ A, so R is a reflexive relation on A.

### II. Symmetric Relation

**Def. :** A relation R on a set A is called a symmetric relation if a R b $\Rightarrow$ b R a where a, b $\in$ A i.e., if (a, b) $\in$ R $\Rightarrow$ (b, a) $\in$ R where a, b $\in$ A.

**For Example :** Let A = {1, 2, 3}

Then A × A = { (1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3,), (3, 2), (3, 3)}

Let R = {(1, 1), (1, 3), (3, 1)}

Clearly, R $\subseteq$ A × A and therefore, R is a relation on A.

Since (x, y) $\in$ R $\Rightarrow$ (y, x) $\in$ R, therefore R is a symmetric relation on A.

### III. Transitive Relation

**Def. :** A relation R on a set A is called a transitive relation if

aRb, b R c $\Rightarrow$ a R c $\forall$ a, b, c, $\in$ R,

i.e, if (a, b) $\in$ R and (b, c) $\in$ R $\Rightarrow$ (a, c) $\in$ R where a, b, c $\in$ A.

**For Example :** For a, b $\in$ N, the set of natural numbers, define a R b if 2a + b=10.

The natural numbers a and b satisfying the relation 2a + b = 0 are given by :

a = 1, b = 8; a = 2, b = 6; a = 3, b = 4; a = 4, b = 2

$\therefore$ R = {(1, 8), (2, 6), (3, 4), (4, 2)}

Since (3, 4) $\in$ R and (4, 2) $\in$ R but (3, 2) $\notin$ R. Therefore R is not a transitive relation.

### IV. Anti-Symmetric Relation

**Def. :** A relation R on a set A is called an anti-symmetric relation if a R b and b R a implies that a = b.

i.e., if $(a, b) \in R$ and $(b, a) \in R \Rightarrow a = b$.

**Note :** Identity relation on a set is symmetric as well as anti-symmetric.

**For Example :** For $a, b \in N$, the set of natural numbers, define a R b if $a \leq b$.

Let $a, b \in N$ such that a R b and b R a.

$\therefore$ $a \leq b$ and $b \leq a$. $\Rightarrow$ $a = b$.

$\therefore$ R is an anti-symmetric relation.

## V. Equivalence Relation

**Def. :** A relation R on a set A is called an equivalence relation if R is reflexive, symmetric and transitive.

**For Example :** Let X be the set of all triangles in a plane.

For any two triangles $\Delta_1$ and $\Delta_2$ in X define $\Delta_1 R \Delta_2$, if $\Delta_1$ and $\Delta_2$ are congruent triangles. Then

**(i) R is Reflexive:** Since each triangle is congruent to itself, so $\Delta R \Delta$ for each $\Delta$ in X.

**(ii) R is Symmetric :** Let $\Delta_1$ and $\Delta_2 \in X$ such that $\Delta_1 R \Delta_2$. Then $\Delta_2$ and $\Delta_1$ are congruent triangles. Hence $\Delta_2 R \Delta_1$.

**(iii) R is Transitive :** Let $\Delta_1, \Delta_2, \Delta_3 \in X$ such that $\Delta_1 R \Delta_2$ and $\Delta_2 R \Delta_3$, i.e., $\Delta_1$, $\Delta_2$ are congruent triangles and so are $\Delta_2$ and $\Delta_3$. This implies that the $\Delta_1$ and $\Delta_3$ are also congruent triangles. Hence $\Delta_1 R \Delta_3$.

So, R is reflexive, symmetric and transitive.

Therefore, R is an equivalence relation on X.

## VI. Partial-Order Relation :

A relation R on a set A is called partial order relation if it is reflexive, anti-symmetric and transitive.

**For Example :** For $a, b \in N$, the relationR defined by a R b if $a \leq b$, is partial-order relation.

## VII. Some other Relations on a Set

**Def. Void Relation :** Since $\phi$ is a subset of $A \times A$, therefore the null set $\phi$ is also a relation in A, called the void relation in a set A.

**Universal relation in a set:** Let A be any set and R be the set $A \times A$. Then R is called the universal relation in A.

**Identity relation in a set :** Let A be any set. Then the relation R defined by

$R = \{(a, a) : \text{for all } a \in A\}$ is called identity relation in A. It is usually denoted by $I_A$.

**Compatible Relation.** A relation R in A is said to be compatible relation if it is reflexive and symmetric.

## VIII. Inverse of a Relation

The inverse of a relation R, denoted by $R^{-1}$, is obtained from R by interchanging the first and second components of each ordered pair of R.

Therefore, $R^{-1} = \{ (a, b) : (b, a) \in R\}$.

If R is a relation from a set A to set B, then $R^{-1}$ is relation from the set B to the set A.

$\therefore$ Domain of $R^{-1}$ = Range of R and Range of $R^{-1}$ = Domain of R.

**For Example :** Let A = {1, 2, 3} and Let R = { (1, 2), (1, 3), (2, 3), (3, 2)}.

Then R is a relation on the set A, since $R \subseteq A \times A$.

$\therefore$ $R^{-1}$ = { (2, 1), (3, 1), (3, 2), (2, 3)}.

**Example 1 :** Give an example of a relation which is reflexive but neither symmetric nor transitive.

**Sol.** Let A = {2, 3, 4}.

Then $A \times A$ = { (2, 2), (2, 3), (2, 4), (3, 2), (3, 3), (3, 4), (4, 2), (4, 3), (4, 4)}

Let R = {(2, 2), (3, 3), (4, 4), (2, 3), (4, 3), (3, 4)}

Since $R \subseteq A \times A$, therefore R is a relation on A.

R is reflexive since $(a, a) \in R \ \forall \ a \in A$.

R is not symmetric since $(2, 3) \in R$ but $(3, 2) \notin R$.

R is not transitive since (2, 3) and $(3, 4) \in R$ but $(2, 4) \notin R$.

Further R is not anti-symmetric since (3, 4) and $(4, 3) \in R$ but $3 \neq 4$.

**Example 2 :** Give an example of a relation which is symmetric but neither reflexive nor transitive.

**Sol.** Let A = {1, 2}.

Then $A \times A$ = { (1, 1), (1, 2), (2, 1), (2, 2)}.

Let R = { (1, 2), (2, 1)}.

Then $R \subseteq A \times A$ and hence R is a relation on the set A.

R is symmetric since $(a, b) \in R \Rightarrow (b, a) \in R$.

R is not reflexive since $1 \in A$ but $(1, 1) \notin A$.

R is not transitive since $(1, 2) \in R$, $(2, 1) \in R$ but $(1, 1) \notin R$

R is not anti-symmetric since $(1, 2) \in R$ and $(2, 1) \in R$ but $1 \neq 2$.

**Example 3 :** The relation $R \subseteq N \times N$ is defined by $(a, b) \in R$ if and only if 5 divides b – a. Show that R is an equivalence relation.

**Sol.** The relation $R \subseteq N \times N$ is defined by $(a, b) \in R$ if and only if 5 divides b – a. This means that R is a relation on N defined by, if $a, b \in N$ then $(a, b) \in R$ if and only if 5 divides b – a.

Let a, b, c belongs to N. Then

(i) $a - a = 0 = 5 \cdot 0$.

$\therefore$ 5 divides a – a.

$\Rightarrow$ $(a, a) \in R$. $\Rightarrow$ R is reflexive.

(ii) Let $(a, b) \in R$.

$\therefore$ 5 divides a – b.

$\Rightarrow$ a – b = 5 n for some $n \in N$. $\Rightarrow$ b – a = 5 (–n).

$\Rightarrow$ 5 divides b – a $\Rightarrow$ $(b, a) \in R$.

$\therefore$ R is symmetric.

(iii) Let (a, b) and $(b, c) \in R$.

$\therefore$ 5 divides a – b and b – c both

$\therefore$ $a - b = 5n_1$ and $b - n^c = 5 n_2$ for some $n_1$ and $n_2 \in N$

$\therefore$ $(a - b) + (b - c) = 5n_1 + 5n_2$ $\Rightarrow a - c = 5 (n_1 + n_2)$

$\Rightarrow$ 5 divides a – c

$\Rightarrow$ $(a, c) \in R$

$\therefore$ R is transitive relation in N.

**Example 4 :** Prove that the intersection of two equivalence relations on a non-empty set is again an equivalence relation on that set.

**Sol.** Suppose that $R_1$ and $R_2$ are two equivalence relations on a non-empty set X.

First we prove that $R_1 \cap R_2$ is an equivalence relation on X.

**(i) $R_1 \cap R_2$ is reflexive :**

Let $a \in X$ arbitrarily.

Then $(a, a) \in R_1$ and $(a, a) \in R_2$, since $R_1$ , $R_2$ both being equivalence relations are reflexive.

So, $(a, a) \in R_1 \cap R_2$

$\Rightarrow$ $R_1 \cap R_2$ is reflexive.

**(ii) $R_1 \cap R_2$ is symmetric :**

Let $a, b \in X$ such that $(a, b) \in R_1 \cap R_2$

$\therefore$ $(a, b) \in R_1$ and $(a, b) \in R_2$

$\Rightarrow$ $(b, a) \in R_1$ and $(b, a) \in R_2$, since $R_1$ and $R_2$ being equivalence relations are also symmetric.

$(b, a) \in R_1 \cap R_2$

$(a, b) \in R_1 \cap R_2$ implies that $(b, a) \in R_1 \cap R_2$.

$\therefore$ $R_1 \cap R_2$ is a symmetric relation.

**(iii) $R_1 \cap R_2$ is transitive :**

Let $a, b, c \in X$ such that $(a, b) \in R_1 \cap R_2$ and $(b, c) \in R_1 \cap R_2$.

$(a, b) \in R_1 \cap R_2$ $\Rightarrow$ $(a, b) \in R_1$ and $(a, b) \in R_2$ ... (i)

$(b, c) \in R_1 \cup R_2$ $\Rightarrow$ $(b, c) \in R_1$ and $(b, c) \in R_2$ ... (ii)

(i) and (ii) $\Rightarrow$ (a, b) and $(b, c) \in R_1$

$\Rightarrow$ $(a, c) \in R_1$, since $R_1$ being an equivalence relation is also trasnsitive.

Similarly, $(a, c) \in R_2$.

$\therefore$ $(a, c) \in R_1 \cap R_2$

So, $R_1 \cap R_2$ in transitive.

Thus $R_1 \cap R_2$ is reflexive, symmetric and also transitive. Thus $R_1 \cap R_2$ is an equivalence relation.

**Example 5 :** If R is an equivalence relation on a set A, then so is $R^{-1}$

**Sol.** Let a, b, c $\in$ A. Then

(i) (a, a) $\in$R, since R being equivalence relation is also a symmetric relation

$\Rightarrow$ (a, a) $\in R^{-1}$ $\Rightarrow$ $R^{-1}$ is reflexive

(ii) Let (a, b) $\in R^{-1}$ $\Rightarrow$ (b, a) $\in$ R

$\Rightarrow$ (a, b) $\in$ R, since R is symmetric

$\Rightarrow$ (b, a) $\in R^{-1}$

$\therefore$ (a, b) $\in R^{-1}$ $\Rightarrow$ (b, a) $\in R^{-1}$.

So, $R^{-1}$ is also symmetric

(iii) Let (a, b) and (b, c) $\in R^{-1}$

$\therefore$ (b, a), (c, b) $\in$ R.

$\Rightarrow$ (c, b), (b, a) $\in$ R.

$\Rightarrow$ (c, a) $\in$ R, since R is transitive.

$\Rightarrow$ (a, c) $\in R^{-1}$

$\therefore$ $R^{-1}$ is also transitive.

$\therefore$ $R^{-1}$ is an equivalence relation.

### 1.3.3 Composition of Relations

**Def. :** Let A, B and C be sets and let R be a relation from A to B and let S be a relation from B to C. That is, R is a subset of A × B and S is a subset of B × C. Then, R and S give rise to a relation from A to C denotes by RoS and defined by

a(RoS) c if for some b $\in$ B we have aRb and bSc

That is RoS = {(a, c) : there exists b $\in$ B for which (a, b) $\in$ R and (b, c) $\in$ S}

The relation RoS is called the composition of R and S ; it is sometimes denoted simply by RS, RoR is denoted by $R^2$, $R^3$ = RoRoR.

**For Example :** Let R and S defined on A be

R = {(1, 1), (3, 1), (3, 4), (4, 2), (4, 3)}

S = {(1, 3), (2, 1), (3, 1), (3, 2), (4, 4)}

Now, RoS = }(1, 3), (3, 3), (3, 4), (4, 1), (4, 2)}

$R^3$ = {(1, 1), (3, 1), (3, 4), (4, 1), (4, 2)}.

### 1.3.4 Closures of a Relation

Let R be a relation in a set A. R may not satisfy particular property like reflexivity, symmetry or transitivity. The new relation, obtained after adding least number of new pairs so that R satisfies particular property, is called closure of R. The types of closures are discussed below :

**Reflexive Closure :** Let R be a relation on A. A reflexive closure of R is the smallest reflexive relation that contains R.

**Symmetric Closure :** Let R be a relation on A which is not symmetric.

$\therefore$ there exists (a, b) $\in$ R but (b, a) $\notin$ R

Now $(b, a) \in R^{-1}$

$\therefore$ to make R symmetric, we add all pairs of $R^{-1}$.

$\therefore$ $R \cup R^{-1}$ is symmetric closure of R.

If R is a relation on A which is not symmetric. Then $R \cup R^{-1}$ is symmetric closure of R.

**Transitive Closure :** Let A be a set and R be a relation on A. The transitive closure of R, denoted by $R^+$, is the smallest relation which contains R as a subset and which is transitive.

**Another Definition :** Let A be a set and R be a relation on A. The relation $R^+ = R \cup R^2 \cup R^3 \ldots\ldots\ldots$ in A is called the transitive closure of R in A.

**Example 6 :** Let R be a relation on a set A = {1, 2, 3} defined by R = {(1, 1), (1, 2), (2, 3)}. Find the reflexive closure of R and symmetric closure of R.

**Sol.** A = {1, 2, 3}

R = {(1, 1), (1, 2), (2, 3)}

$R^{-1}$ = { (1, 1), (2, 1), (3, 2)}

$R \cup R^{-1}$ = {(1, 1), (1, 2), (2, 1), (2, 3), (3, 2)}

Reflexive closure of R is {(1, 1), (2, 2), (3, 3), (1, 2), (2, 3)}

Symmetric closure of R is $R \cup R^{-1}$ = {(1, 1), (1, 2), (2, 1), (2, 3), (3, 2)}.

**Example 7 :** Let R be a relation on set A = {1, 2, 3, 4} defined by

R = {(1, 2), (2, 3), (3, 4), (2, 1)}. Find transitive closure of R.

**Sol.** A = {1, 2, 3, 4}

R = {(1, 2), (2, 3), (3, 4), (2, 1)}

$$\therefore \quad M = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{where m is matrix of R}$$

, where M is matrix of R.

$$M^2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$M^3 = M^2M = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$M^4 = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\therefore \quad M^4 = M + M^2 + M^3 + M^4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ $R^+$ = {(1, 1), (1, 2), (1, 3), (1, 4), (2, 1), (2, 2), (2, 3), (2, 4), (3, 4)}
which is transitive closure of R.

## 1.3.5 Equivalence Class

Consider, an equivalence relation R on a set A. The equivalence class of an element $a \in A$, is the set of elements of A to which element a is related. It is denoted by [a].

**For Example :** Let A = {4, 5, 6, 7} and R = {(4, 4), (5, 5), (6, 6), (7, 7), (4, 6), (6, 4)} be an equivalence relation on A.

Now, equivalence classes are as follows :

[4] = [6] = {4, 6}

[5] = {5}

[7] = {7}

**Results :** (i) Suppose that R is an equivalence relation on a set X,

Then (I) $a \in [a] \ \forall \ a \in X$.

(II) $a \in [b]$ iff $[a] = [b] \ \forall \ a, b \in X$

(III) $[a] = [b]$ or $[a] \cap [b] = \phi \ \forall \ a, b \in X$, i.e. any two equivalence classes are either disjoint or identical. Proof : Try Yourself.

(ii) The distinct equivalence classes of an equivalence relation on a set form a partition of that set.

## 1.3.6 Representing Relations

In order to represent a relation, there are numerous ways such as matrix representation, graphical representation, arrow diagram, diagraph or directed graph and Hasse diagram. All these may be clearly understood from the following examples:

**Example 8 :** If A = {1, 2, 3, 4} and B = {x, y, z}. Let R be the following relation from A to B : R = (1, y), (1, z), (3, y), (4, x), (4, z).

(a) Determine the matrix of the relation

(b) Draw the arrow diagram of R

**Sol.** (a) From the fig. 1 Observe that rows of the matrix are labeled by the elements of A and the columns by the elements of B. Also observe that entry in the matrix corresponding to $a \in A$ and $b \in B$ is 1 if a is related to b and 0 otherwise.

$$\begin{array}{c c} & \begin{array}{ccc} x & y & z \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \end{array}$$

**Fig. 1**

(b) From fig. 2, Observe that there is an arrow from $a \in A$ to $b \in B$ iff a is related to b i.e. iff $(a, b) \in R$.

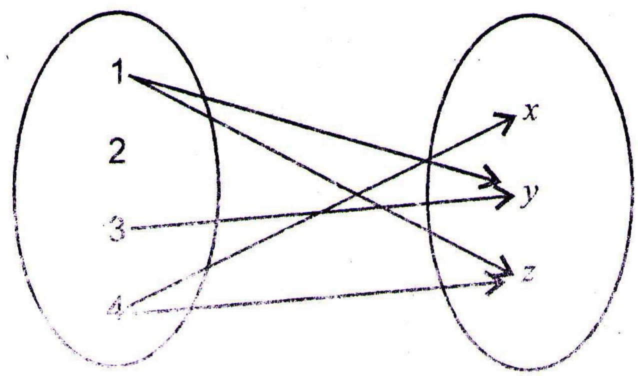



**Fig. 2**

**Example 9 :** If A = {1, 2, 3, 4}, consider the following relation in A

R = {(1, 1), (2, 2), (2, 3), (3, 2), (4, 2), (4, 4)}

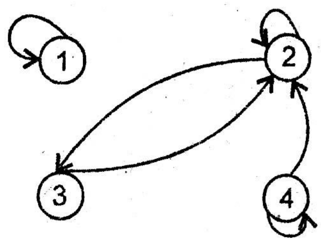
(a) Draw its directed graph.

(b) Is R (i) reflexive, (ii) symmetric (iii) transitive or (iv) antisymmetric

(c) $R^2$ = RoR

Before giving the solution, we define the directed graph or cligraph as :- Draw a small circle for each elemet of A. These circles are called vertices. Draw an arrow, called a edge, from vertex $a_i$ to vertex aj iff ai Raj. The resulting pictorial representation of R is called a directed graph or digraph.

**Sol.** (a)




(b) (i) R is not reflexive $3 \in A$ but $3 \not R\ 3$ i.e. $(3, 3) \notin R$.

(ii) R is not symmetric because 4 R 2 but $2 \not R\ 4$

i.e. $(4, 2) \in R$ but $(2, 4) \notin R$.

(iii) R is not transitive because 4 R 2 and 2 R 3 but $4 \not R\ 3$ i.e. $(4, 2) \in R$ and $(2, 3) \in R$ but $(4, 3) \notin R$.

(iv) R is not anti-symmetric because 2 R 3 and 3 R 2 but $2 \neq 3$.

(c) For each pair $(a, b) \in R$, find all $(b, c) \in R$ since $(a, c) \in R^2$

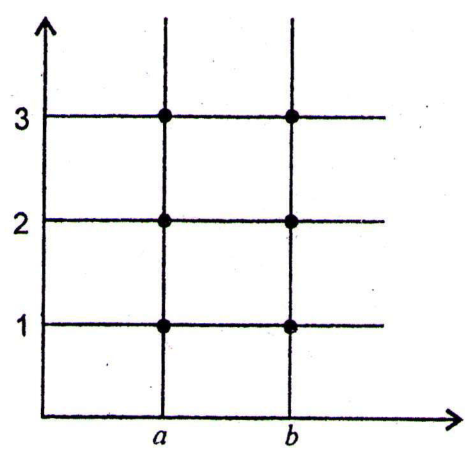
$R^2 = \{(1, 1), (2, 2), (2, 3), (3, 2), (3, 3), (4, 2), (4, 3), (4, 4)\}$.

**Example 10 :** Let $A = \{1, 2, 3\}$ and $B = \{a, b\}$. Represent $A \times B$ graphically. What is $|A \times B|$ ?

**Sol.** $A \times B = \{(1, a), (1, b), (2, a), (2, b), (3, a), (3, b)\}$

Graphically $A \times B$ is shown below :

($|A \times B|$ represents the order of $A \times B$ i.e. No. of elements in $A \times B$).

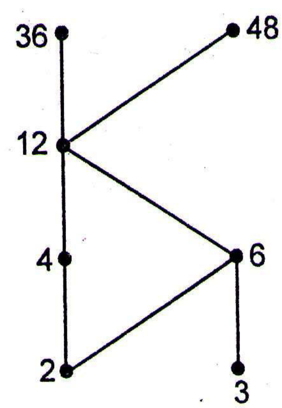



$|A \times B| = |A| . |B| = 3.2 = 6.$

Now, we define **Hasse diagram** as :

**Def. :** The Hasse diagram of a partially ordered relation. R defined on a set X is a directed graph whose vertices are the elements of X and there is an undirected edge from a to b whenever $(a, b) \in R$ (Instead of drawing an arrow from a to b, we some times place b higher than a and draw a line between them). An arrow from a vertex to itself is drawn whenever $(a, a) \in R$.

**For Example :** Let B = {2, 3, 4, 6, 12, 36, 48} and S be the relation/"divide" on B. Then, Hasse diagram of S is




## 1.3.7 Introduction to Functions

Firstly, we define a function as :

**Def. Function :** Let X and Y be two non-empty sets. A subset f of $X \times Y$ is called a function from X to Y if for each $x \in X$, there exists a unique $y \in Y$ such that $(x, y) \in f$ or $f(x) = y$. It may also be defined as a rule f which associates each element of X with a unique element of Y. It is denoted by $f : X \to Y$ or $X \xrightarrow{f} Y$. Here, the set X is called the domain of f and is written as $D_f = X$. The set Y is called co-domain of f. If an element $y \in Y$ is associated with an element x of X under the rule f, then y is called the image of x under the rule f, denoted by f(x). The set consisting of images of all the elements of X under f is called Image set or Range of f and is written as $R_f$. Mathematically, $R_f = \{y : y = f(x) \text{ where } x \in X\} = f(X)$ clearly, $f(X) \subset Y$.

**Remarks :** (i) Functions are also called mappings or transformations.

(ii) To every $x \in X$, $\exists$ a unique $y \in Y$ such that $y = f(x)$. The unique element $y \in Y$ is also called the value of f at x and is denoted by f (x).

(iii) Different elements of X may be associated with the same element of Y.

(iv) There may be elements of Y which are not associated with any element of X.

**For Example :** (i) The rule shown in the figure 4.7 is not a function as each element of X is not asociated. Here $5 \in X$ has no image in Y.

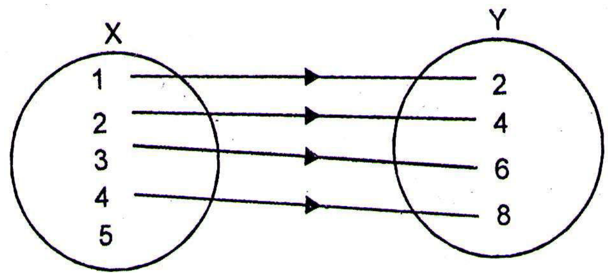



**Fig. 4.7**

(ii) The rule shown in the figure 4.8 is not a function as I $\in$ X is associated with more than one element namely a and b of Y.

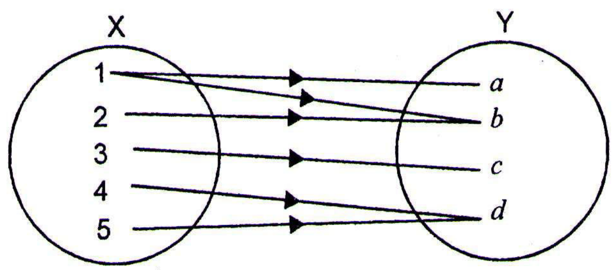



**Fig. 4.8**

(iii) The rule shown in the figure 4.9 is a function as each element of X is associated with a unique element of Y.

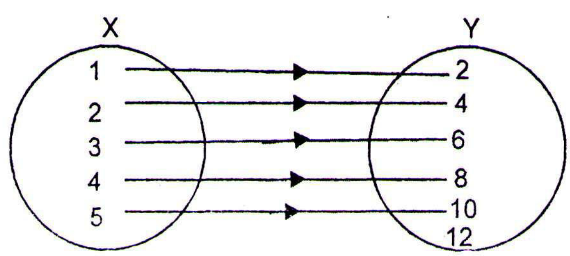



**Fig. 4.9**

## 1.3.8 One-One and Onto Functions

**Def. One-One function or Injective function :** A function f from X to Y is said to be one-one (abbreviated 1-1) iff

$x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)\ \forall\ x_1, x_2 \in X$, or equivalently

$f(x_1) = f(x_2) \Rightarrow x_1 = x_2\ \forall\ x_1, x_2 \in X$.

In other words, if different elements of X under the rule f have different images in Y, then f is called one-one function. A function which is not 1-1 is called many-one function.

**For Example :**

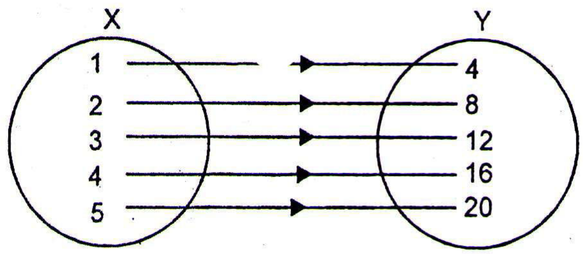



**Fig. 4.10**

## Def. Onto function or Surjective Function :

**Def.** A function from f from X to Y is called onto iff every element of Y is an image of at least one element of X. Otherwise f is called an into mapping.

**Note.** In the case of onto function, $R_f = Y$, while in the case of into function $R_f$ is a proper subset of Y.

**For Example :** (i) The function f depicted in the below diagram is one-one and onto function.

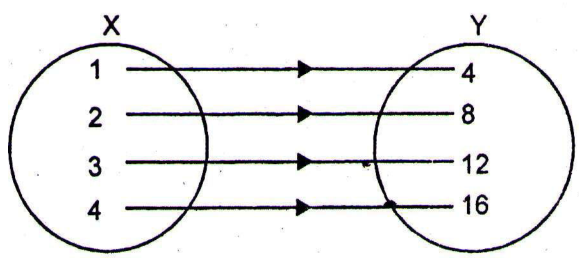



**Fig. 4.11**

(ii) Let X = {1, 2, 3, 4, 5, 6}, Y = {2} (It is onto but not one-one)

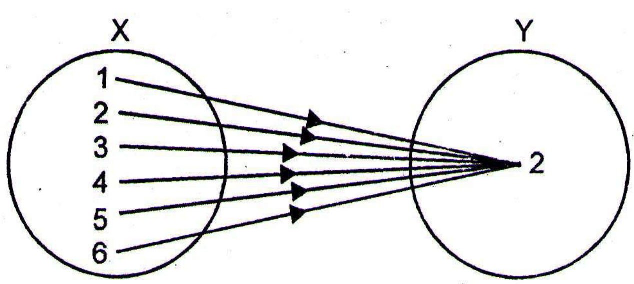



**Fig. 4.12**

**Def. Bijective function :** A function which is one-one and onto is called bijective function. It is also called **one-one correspondence.**

**For Example :** The function shown in fig. 4.11 is a bijective function.

## 1.3.9 Types of Functions

There are various types of functions as discussed below :

### I. Real valued function on real variables

Let X, Y be two non-empty subsets of real numbers. Then, every function f from X to Y is called a real valued function on real variables.

### II. Equal functions

Two real valued functions f and g are said to be equal iff $D_f = D_g$ and

$f(x) = g(x)\ \forall\ x \in D_f$. We write it as f = g.

### III. Constant Function

A function $f : X \to Y$ is called a constant function if $f(x) = k$ for every $x \in X$ and Here $k \in Y$ is fixed.

Function shown in figure 4.12 is a constant function.

### IV. Identity Mapping

Let $I_x : X \to X$ be defined by, $I_x(x) = x\ \forall\ x \in X$.

Then $I_x$ is called the identity mapping on X.

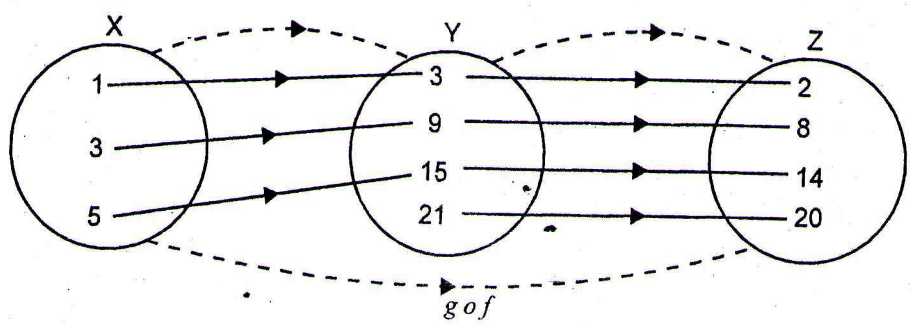
### V. Inverse Mapping

Let $f : X \to Y$ be a one-one onto mapping. Then the mapping $f^{-1} : Y \to X$ which associates to each element $y \in Y$, the unique element $x \in X$ such that $f(x) = y$ is called the inverse map of f.

## 1.3.10 Composition of Functions

**Def. :** Let f be a function with domain X and range in Y and let g be a function with domain Y and range in Z. The function with domain X and range in Z which maps an element $x \in X$, into $g(f(x))$, is called the composite of the functions f and g and is written as g of.

**For Example :** Let X = {1, 3, 5}, Y = {3, 9, 15, 21}, Z = {2, 8, 14, 20}



Let f be a function from X to Y and g be a function from Y to Z such that

$f = \{(1, 3), (3, 9), (5, 15)\}$, $g = \{(3, 2), (9, 8), (15, 14), (21, 20)\}$

then, $gof = \{(1, 2), (3, 8), (5, 14)\}$.

**Note :** (i) gof is defined only when $R_f \subset D_g$.

(ii) It is possible that one of fog and gof may be defined while the other may not be defined.

(iii) gof and fog both may be defined but may not be equal.

(iv) Let f, g, h be three functions and $\alpha$ be a real number, then

(a) (fog) oh = fo (goh) (Associative Law)

(b) fo (g+h) = fog + foh (Distributive Law)

(c) $(\alpha f)$ og = $\alpha$. (fog) (Scalar multiplication)

**Art 1 :** If $f : A \to B$ and $g : B \to C$ are both one-one and onto maps i.e. bijective maps, then gof is also bijective map.

**Proof :** Since $f : A \to B$ and $g : B \to C$ are maps, therefore gof is also a map from A to C. Let $x_1, x_2 \in A$ such that

**One–One :** $(gof)\ (x)_1 = (gof)\ x_2$

$\Rightarrow$ $g(f(x_1)) = g(f(x_2))$

$\Rightarrow$ $f(x_1) = f(x_2)$, since g is one-one

$\Rightarrow$ $x_1 = x_2$ since f is one-one

$\therefore$ gof is a one-one map

**Onto :** Since f, g are onto

Let $c \in C$ be any element, then $\exists\ b \in B$ such that

$g(b) = c$ ($\because$ g is onto)

Again for this $b \in B$, $\exists$ some $a \in A$ such that

$f(a) = b$ ($\therefore$ f is onto)

$\therefore$ $gof(a) = g(f(a)) = g(b) = c$

Thus for $c \in C$, $\exists\ a \in A$ such that $gof(a) = c$

Hence $gof : A \to C$ is onto.

**Art 2 :** If $f : A \to B$ and $g : B \to C$ are two maps such that $gof : A \to C$ is both one-one and onto map, then f is one-one and g is onto.

**Proof :** Since $f : A \to B$, $g : B \to C$ are maps

$\therefore$ $gof : A \to C$ is a map. Also gof is given to be one-one map.

**f is one-one :** If possible, suppose that f is not one-one and g is one-one.

$\therefore$ $\exists\ x_1, x_2 \in A$ such that $x_1 \neq x_2$ but $f(x_1) = f(x_2)$.

But $f(x_2) = f(x_2) \Rightarrow g(f(x_1)) = g(f(x_2))$ [$\because$ g is supposed to be one-one]

$\Rightarrow$ $gof(x_1) = gof(x_2)$

$\therefore$ $x_1, x_2 \in A$ such that $x_1 \neq x_2$, but (gof) $(x_1)$ = (gof) $(x_2)$

$\therefore$ gof is not one-one, which is against the given hypothesis that gof is one-one.

Thus our supposition is wrong and f is one-one but g is not one-one.

We, now give an example to illustrate that if gof is one-one, then g may not be one-one.

Let A = {1, 2}, B = {4, 5, 6}, C = {7, 8, 9, 10}

Let f = {(1, 4), (2, 6)} and g = {(4, 7), (5, 8), (6, 8)}

then f and g are functions from A to B and from B to C respectively.

Have $R_f = \{5, 6\} \subseteq D_g = \{4, 5, 6\}$

$\therefore$ $R_f \subseteq D_g$

$\Rightarrow$ gof is defined and $D_{gof} = D_f = A = \{1, 2\}$

gof (1) = g (f(1)) = g(4) = 7 and gof (2) = g (f(2)) = g (6) = 8

$\therefore$ gof = {(1, 7), (1, 8)}

Here, gof is one-one map since different elements of A have different image

But g is not one-one since g (5) = g (6) = 8, but $5 \neq 6$.

Since $f : A \to B$ and $g : B \to C$ are maps, so gof is a map from A to C. We are given that gof : $A \to C$ is onto. We now prove that g is onto

Let $z \in C$.

**g is onto :** Since gof : $A \to C$ is onto, so $\exists\, x \in A$ such that gof (x) = z

$\Rightarrow$ g (f (x)) = z $\Rightarrow$ g (y) = z where y = f (x)

Since $x \in A$ and f is map from A to B

Therefore $f(x) \in B \Rightarrow y \in B$

for given $z \in C$, we have determined $y \in B$ such that g (y) = z

$\therefore$ $g : B \to C$ is onto

Now, we show by an example that if gof is onto, then f may not be onto

Let A = {1, 2}, B = {4, 5, 6}, C = {7}

Let f = {(1, 4), (2, 6)} and g = {(4, 7), (5, 7), (6, 7)}

Then f is a function from A to B and g is a function from B to C

$\therefore$ gof is a function from A to C such that gof = {(1, 7), (2, 7)}

Here, g is onto. But f is not onto since 5 belonging to B has no pre-image in A under the map f but $g : B \to C$ is onto.

## 1.3.11 Invertible Function

**Def. :** A function f defined from X to Y is said to be invertible if three exists a function g from Y to X such that gof = $I_X$ and fog = $I_Y$, where $I_X$ is an identity mapping on X and $I_Y$ is an identity mapping on Y.

**Note :** f and g are called inverse of each other.

**Art 3 :** Let $f : X \to Y$. Then fo $I_x$ = f = $I_Y$ of.

**Proof :** Let x be any element of X and let f (x) = y, $y \in Y$

Since $f : X \to Y$ and $I_Y : Y \to Y$

$\therefore$ $I_X of : X \to Y$

Now $(I_Y of)(x) = I_Y(f(x)) = I_Y(y) = y = f(x) \ \forall\ x \in X$

$\therefore$ $I_Y of = f$

Again $I_X : X \to X$ and $f : X \to Y$

$\therefore$ $fo\ I_X : X \to Y$

Now $(fo\ I_X)(x) = f(I_X(X)) = f(x) \ \forall\ x \in X$

$\therefore$ $fo\ I_X = f$.

**Art 4 :** If $f : X \to Y$ is invertible, then its inverse is unique.

**Proof :** Let $g : Y \to X$ and $h : Y \to X$ be two inverse functions of $f : X \to Y$

$\therefore$ $fog = I_Y$, $gof = I_X$ and $foh = I_Y$, $hof = I_X$

Now $g(y) = g(I_Y(y)) = g\{(foh)(y)\} = g\{f(h(y))\}$

$= (gof)(h(y)) = I_X(h(y))$

$\therefore$ $g(y) = h(y) \ \forall\ h(y) \in X \Rightarrow g(y) = h(y) \ \forall\ y \in Y$

$\therefore$ $g = h$

$\therefore$ inverse of function f is unique.

**Note.** (1) Inverse of f, if it exists is denoted by $f^{-1}$.

(2) $f^{-1} of = I_X$ and $f\ of^{-1} = I_Y$ where $f : X \to Y$ is an invertible function.

**Art 5 :** A function $f : X \to Y$ is invertible iff f is one-one and onto.

**Proof.** (i) Assume that $f : X \to Y$ is invertible

$\therefore$ $\exists$ a function $g : Y \to X$ such that $fog = I_Y$ and $gof = I_X$

We will prove that f is one-one and onto.

**To prove that f is one-one**

Let $x_1, x_2 \in X$ and $f(x_1) = f(x_2)$

$\Rightarrow$ $g(f(x_1)) = g(f(x_2))$ $\Rightarrow$ $(gof)(x_1) = (gof)(x_2)$

$\Rightarrow$ $I_X(x_1) = I_X(x_2)$ $\Rightarrow$ $x_1 = x_2$

$\therefore$ $f(x_1) = f(x_2)$ $\Rightarrow$ $x_1 = x_2 \ \forall\ x_1, x_2 \in X$

$\therefore$ f is one-one.

**To prove f is onto**

To each $y \in Y$, there exists $x \in X$ such that $g(y) = x$.

$\Rightarrow$ $f(g(y)) = f(x)$ $\Rightarrow$ $(fog)(y) = f(x)$

$\Rightarrow$ $I_Y(y) = f(x)$ $\Rightarrow$ $y = f(x)$

$\therefore$ f is onto.

(ii) Assume that $f : X \to Y$ is one-one and onto. We have to show that f is invertible.

Since f is one-one and onto

$\therefore$ to each $y \in Y$, there exists one and only one $x \in X$ such that $f(x) = y$.

$\therefore$ we can define a function $g : Y \to X$ such that $g(y) = x$ iff $f(x) = y$

Now (gof) (x) = g (f (x)) = g(y) = x, $\forall$ x $\in$X

$\therefore$ gof = $I_x$

Again (fog) (y) = f(g (y)) = f(x) = y, $\forall$ y $\in$ Y

$\therefore$ fog = $I_y$

$\therefore$ f is invertible.

**Art 6 :** If a function f : X $\to$ Y be one-one and onto then, $f^{-1}$ is also one-one and onto.

**Proof :** $\because$ f : X $\to$ Y is one-one and onto

$\therefore$ $f^{-1} : Y \to X$ exists and $f^{-1} of = I_X$, $fof^{-1} = I_Y$

**To prove $f^{-1}$ is one-one**

Let $y_1 \in Y$, $y_2 \in Y$.

Now $f^{-1}(y_1) = f^{-1}(y_2)$

$\Rightarrow$ $f(f^{-1}(y_1)) = f(f^{-1}(y_2))$

$\Rightarrow$ $(fof^{-1})(y_1) = (fof^{-1})(y_2)$

$\Rightarrow$ $I_Y(y_1) = I_Y(y_2)$ $\Rightarrow$ $y_1 = y_2$

$\therefore$ $f^{-1}(y_1) = f^{-1}(y_2)$ $\Rightarrow$ $y_1 = y_2 \ \forall\ y_1, y_2 \in Y$

$\therefore$ $f^{-1}$ is one-one.

**To prove $f^{-1}$ is onto**

To each x $\in$ X, there exists y $\in$ Y such that y = f(x)

$\Rightarrow$ $f^{-1}(y) = f^{-1}(f(x))$ $\Rightarrow$ $f^{-1}(y) = (f^{-1} of)(x)$

$\Rightarrow$ $f^{-1}(y) = I_X(x)$ $\Rightarrow$ $f^{-1}(y) = x$

$\therefore$ $f^{-1}$ is onto

**Cor.** Since, $f^{-1}$ is invertible and its inverse is f.

$\therefore$ $(f^{-1})^{-1} = f$.

**Art 7 :** Let f : X $\to$ Y and g : Y $\to$ Z and let f, g be one-one onto. Then gof : X $\to$ Z is also one-one onto and $(gof)^{-1} = f^{-1} og^{-1}$.

**Proof :** Do Yourself.

## 1.3.12 Floor and Ceiling Functions

For any real number x, the floor function of x means the greatest integer which is less than or equal to x.

It is denoted by [x].

**For Example :** [2.58] = 2, [–4.4] = 5, [2] = 2

For any real number x, the ceiling function of x means the least integer which is greater than or equal to x. It is denoted by [x].

**For Example :** [2.58] = 3, [–4.4] = –4, [2] = 2.

For any real number x, the integer function of x converts x into an integer by deleting the fractional part of x. It is denoted by INT (x).

**For Example :** INT (2.44) = 2, INT (–4.44) = –4.

**Note :** (i) If x is an integer, then [x] = [x]. Otherwise [x] + 1 = [x]

(ii) $[x] = n \Rightarrow n \le x < n + 1$ and $[x] = n \Rightarrow n-1 < x \le n$

(iii) INT (x) = [x] if x is positive and INT (x) = [x] if x is negative.

## Some Important Examples

**Example 11 :** Let a function f : **R** → **R** be defined by f (x) = 2x + 3 ∀ x ∈ R. Prove that f is one-one and onto.

**Sol.** Let $x_1, x_2 \in R$ such that $f(x_1) = f(x_2)$

∴ $2x_1 + 3 = 2x_2 + 3 \Rightarrow x_1 = x_2$

∴ f is one-one map

Let $y \in R$. Let $y = f(x_0)$

Then $2x_0 + 3 = y \Rightarrow x_0 = \dfrac{y-3}{2}$

Since $y \in R$, so $\dfrac{y-3}{2} \in R$ i.e., $x_0 \in R$

$$f(x_0) = 2\,x_0 + 2 = 2\,\frac{y-3}{2} + 3 = y$$

Therefore for each $y \in R$, there exists $x_0 \in R$ such that $f(x_0) = y$.

∴ f is onto.

**Example 12 :** If $f(x) = \dfrac{1}{1-x}$, then what is f [f {f(x)}] ?

**Sol.** Here $f(x) = \dfrac{1}{1-x}$

$$\therefore \qquad f\{f(x)\} = \frac{1}{1-f(x)} = \frac{1}{1-\dfrac{1}{1-x}} = \frac{1-x}{1-x-1} = \frac{1-x}{-x}$$

$$\therefore \qquad f\,[f\{f(x)\}] = \frac{1-f(x)}{-f(x)} = \frac{1-\dfrac{1}{1-x}}{\dfrac{-1}{1-x}} = \frac{1-x-1}{-1} = \frac{-x}{-1}$$

⇒ f [f {f(x)}] = x.

**Example 13 :** Let f and g be two functions from **IR** → **R** defined by f (x) = $x^2 + 3x + 2$ and g (x) = 4x – 1. Find fog and gof. Also calculate (gof) (–1) and fog (–1). Is composition commutative or not ?

**Sol.** gof is a map from A to C defined by gof (x) = g (f (x)) $\forall$ x $\in$ A.

Here, f : $\mathbf{R} \to \mathbf{R}$ is defined by $f(x) = x^2 + 3x + 2$, $\forall$ x $\in$ R,

and g : $\mathbf{R} \to \mathbf{R}$ is defined by $g(x) = 4x - 1$, $\forall$ x $\in$ R

$\therefore$ fog : $\mathbf{R} \to \mathbf{R}$ is defined by

$$fog(x) = f(g(x)) = f(4x + 1) = (4x - 1)^2 + 3(4x - 1) + 2$$

$$= 16x^2 - 8x + 1 + 12x - 3 + 2$$

i.e., $(fog)\ x = 16x^2 + 4x.$ ... (1)

gof : $\mathbf{R} \to \mathbf{R}$ is defined by

$$gof = g(f(x)) = g(x^2 + 3x + 2) = 4(x^2 + 3x + 2) - 1$$

$$= 4x^2 + 12x + 8 - 1$$

$$gof(x) = 4x^2 + 12x + 7 \quad \text{... (2)}$$

By (1), $(fog)(-1) = 16(-1)^2 + 4(-1) = 16 - 4 = 12$

By (2), $(gof)(-1) = 4(-1)^2 + |\ 12(-1) + 7 = -1$

$\therefore$ $fog(-1) \neq gof(-1)$.

Hence composition of maps is not commutative.

**Example 14 :** Is $f(x) = \dfrac{x-1}{x+1}$ invertible in its domain ? If so, find $f^{-1}$. Further verify that $(fof^{-1})(x) = x$.

**Sol.** Here $f(x) = \dfrac{x-1}{x+1}$

$D_f$ = set of all reals except – 1

$R_f$ = set of all reals except 1

Let $x_1, x_2 \in D_f$ and $f(x_1) = f(x_2)$

$$\Rightarrow \quad \frac{x_1 - 1}{x_1 + 1} = \frac{x_2 - 1}{x_2 + 1} \Rightarrow x_1x_2 - x_2 + x_1 - 1 = x_1x_2 + x_2 - x_1 - 1$$

$\Rightarrow$ $2x_1 = 2x_2$ $\Rightarrow$ $x_1 = x_2$

$\therefore$ $f(x_1) = f(x_2)$ $\Rightarrow$ $x_1 = x_2$

$\therefore$ f (x) is 1 – 1 in $D_f$.

$$\forall\ y \in R_f\ \exists\ x = \frac{1+y}{1-y} \in D_f\ (\text{where } y \neq 1)$$

$$\text{s.t. } f(x) = f\left(\frac{1+y}{1-y}\right) = \frac{\frac{1+y}{1-y}-1}{\frac{1+y}{1-y}+1} = \frac{\frac{1+y-1+y}{1-y}}{\frac{1+y-1-y}{1-y}} = \frac{2y}{2} = y$$

$\therefore$ the mapping fis onto

$\therefore$ f is both 1–1 and onto $\Rightarrow f^{-1}$ exists

Now to find $f^{-1}$, Let $y = f(x) = \frac{x-1}{x+1}$

$\therefore$ $xy + y = x - 1 \Rightarrow x - xy = y + 1$

$\Rightarrow$ $x(1-y) = 1 + y \Rightarrow x = \frac{1+y}{1-y}$.

$\therefore$ $f^{-1}(y) = \frac{1+y}{1-y} \Rightarrow f^{-1}(x) = \frac{1+x}{1-x}$

and $D_{f^{-1}} =$ Set of all reals except 1

Verification : $(fof^{-1})(x) = f(f^{-1}(x)) = \frac{f^{-1}(x)-1}{f^{-1}(x)+1} = \frac{\frac{1+x}{1-x}-1}{\frac{1+x}{1-x}+1} = \frac{\frac{1+x-1+x}{1-x}}{\frac{1+x-1-x}{1-x}} = \frac{2x}{2} = x$

$(fof^{-1})(x) = x$.

**Example 15 :** Prove that function $f: C \to R$, defined by $f(z) = |z|$ is neither one-one nor onto.

**Sol.** Here, $f : C \to R$ defined by $f(z) = |z|$

Let $z_1 = 2 + 3i$, $z_2 = 2 - 3i \Rightarrow z_1 \neq z_2$

$f(z_1) = |z_1| = \sqrt{4+9} = \sqrt{13}$ $[z = x + iy, |z| = \sqrt{x^2 + y^2}]$

$f(z_2) = |z_2| = \sqrt{4+9} = \sqrt{13}$

Here $f(z_1) = f(z_2)$. But $z_1 \neq z_2$.

so, f is not one-one.

**Onto :** again let $-3 \in R$.

But there does not exist any complex number such that $f(z) = -3$.

So, f is not onto.

**Example 16 :** Let X = Y = Z = R and let $f : X \to Y$ and $g : Y \to Z$ are such that $f(x) = 2x + 1$ and $g(y) = y/3$. Verify that $(gof)^{-1} = f^{-1}og^{-1}$.

**Sol.** Here $f : X \to Y$ is defined by $f(x) = 2x + 1$ and $g : Y \to Z$ is defined by $g(y) = \frac{y}{3}$

L.H.S. : $gof(x) = g(f(x)) = g(2x+1) = \frac{2x+1}{3}$

Now we find $(gof)^{-1}$, Let gof (x) = y

then, $y = \frac{2x+1}{3} \Rightarrow x = \frac{3y-1}{2}$

$\Rightarrow \quad (gof)^{-1}y = \frac{3y-1}{2}$ or $(gof)^{-1}x = \frac{3x-1}{2}$

R.H.S. : f (x) = 2x + 1

$\Rightarrow \quad y = 2x+1 \Rightarrow x = \frac{y-1}{2}$

$\Rightarrow \quad f^{-1}(y) = \frac{y-1}{2}$ or $f^{-1}(x) = \frac{x-1}{2}$

Again, $g(y) = \frac{y}{3} \Rightarrow x = \frac{y}{3} \Rightarrow y = 3x$

$\Rightarrow \quad g^{-1}(x) = 3x$ or $g^{-1}(y) = 3y$

Now $\quad f^{-1}og^{-1}(x) = f^{-1}(g^{-1}(x)) = f^{-1}(3x) = \frac{3x-1}{2}$

so $\quad (gof)^{-1} = f^{-1} og^{-1}$.

**Example 17:** Let f and g be functions from **R** to **R** defined by f (x) = [x] and g(x) = |x|. Determine whether fog = gof.

**Sol.** Given f (x) = [x] and g(x) = |x|

fog (x) = f [g (x)] = f (|x|) = [|x|]

gof (x) = g [f(x)] = g ([x]) = |[x]|

Now fog ≠ gof.

As fog (–3.2) = f[g (–3.2)] = f (3.2) = 3

gof (–3.2) = g [f (–3.2)] = g (–4) = 4.

### 1.3.13 Self Check Exercise

1. In N × N, show that the relation defined by (a, b) R (c, d) if ad = bc is an equivalence relation.
2. Show that $R_1 \cup R_2$ may not be an equivalence relation on a set X if $R_1$, $R_2$ are equivalence relations on X.
3. How many relations are possible from a set A of m elements of another set B of n elements ? Why ?
4. Let R and S be the relations on X = {a, b, c} defined by
   R = {a, b), (a, c), (b, a)}, S = {(a, c), (b, a), (b, b), (c, a)}
   (i) Find $M_R$ and $M_S$ (ii) Find RoS (iii) Find SoR
5. Let A = {1, 2, 3, 4} and relation on it R = {(a, b) : | a – b| = 2}. Find transitive closure of R.
6. Let X = {1, 2, 3, 4} and R = {(x, y) : x > y}. Draw the diagraph and matrix of R.
7. R is a relation on set of positive integers s.t. R = {(a, b) : a – b is an odd integer}. Is R an equivalence relation ?
8. Prove that a function f : **IR** → **R**, defined by f (x) = $x^3$ is one-one onto.
9. Is function f : **IR** → **R** defined by $f(x) = \frac{1}{x}$ is bijective in its domain.
10. If f(x) = $x^2 - 1$, g (x) = 3x + 1, then describe the following functions :
    (i) gof (ii) fog (iii) gog (iv) fof
11. If y = f(x) = $\frac{x+2}{x-1}$, then show that x = f(y).
12. Let f : **IR** → **R** and g : **IR** → **IR** be real valued functions defined by
    f (x) = $2x^3 - 1$, x ∈ R and $g(x) = \left[\frac{1}{2}(x+1)\right]^{\frac{1}{3}}, x \in \mathbf{IR}$. Show that f and g are bijective and each is inverse of other.
13. Give an example of a map which is
    (i) one-one but not onto (ii) onto but not one-one.

### Suggested Readings :


1. Dr. Babu Ram, Discrete Mathematics
2. C.L. Liu, Elements of Discrete Mathematics (Second Edition), McGraw Hill, International Edition, Computer Science Series, 1986.
3. Discrete Mathematics, S. Series.
4. Kenneth H. Rosen, Discrete Mathematics and its Applications, McGraw Hill Fifth Ed. 2003.

**LESSON NO. 2.1** **Author : Dr. Chanchal**

# GRAPH THEORY–I

**Structure :**



## I. Objectives

The prime objective of this unit is to define and discuss various components of graph theory along with suitable examples.

## II. Introduction

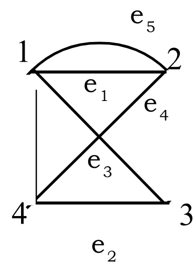
Graph theory is employed in many areas of computer science such as switching theory, logical design, artificial intelligence, formal languages, computer graphics etc. On account of diversity of its application, it is useful to develop and study the subject in abstract form and then import its results. Firstly, we may define a graph as

**Def. Graph :** A graph (or undirected graph) is a diagram consisting of a collection of vertices together with edges joining certain pair of these vertices. Mathematically, we can write a graph G as G = [V(G), E(G)] where V(G) and E(G) are sets defined as V(G) is the vertex set of the graph G, and $E(G) \subseteq V(G) \times V(G)$, a relation on V(G), called edge set of G. Each element e of E(G) is assigned an unordered pair of vertices (a, b), where a and b are end vertices of e.

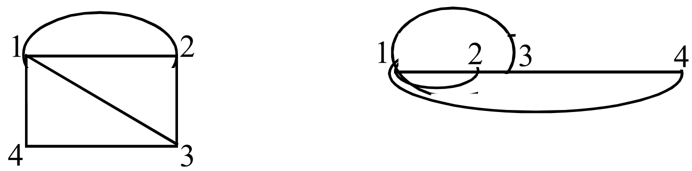
A graph G (or undirected graph) consists of a finite set V of objects called vertices, a finite set E of objects called edges ad a functio Y, that assigns to each edge a subset {v, w} where v and w are vertices (and may be the same). Mathematically, we can write, G=(V,E,Y) or simply G=(V,E). If e is a edge and Y(e)={v,w}, then e is an edge between v and w and that e is determined by v and w. Moreover, v and w are called the end points of 'e'.

e.g. :- Let V = {1,2,3,4} and $E=\{e_1, e_2, e_3, e_4, e_5)$

Let Y be defined by $Y(e_1) = Y(e_5) = \{1,2\}$
$Y(e_2) = \{4,3\}$, $Y(e_3) = \{1,3\}$, $Y(e_4) = \{2,4\}$
Then, (G,E,Y) is a graph.




In the pictorial representation of a graph, the connections are the most important information and generally, a no. of different pictures may represent the same graph. e.g.




**Note :** It does not matter whether the joining of the two vertices in a graph is a straight line or a curve, longer or shorter.

**Def. Directed Graph :** It may be defined as a graph in which each element 'e' of E(G) is assigned an ordered pair of vertices (a, b) along with arrow starting from 'a' to 'b', where 'a' is called initial vertex and 'b' is called terminal vertex of the edge e. The graphs directed and undirected are shown in the following figures.

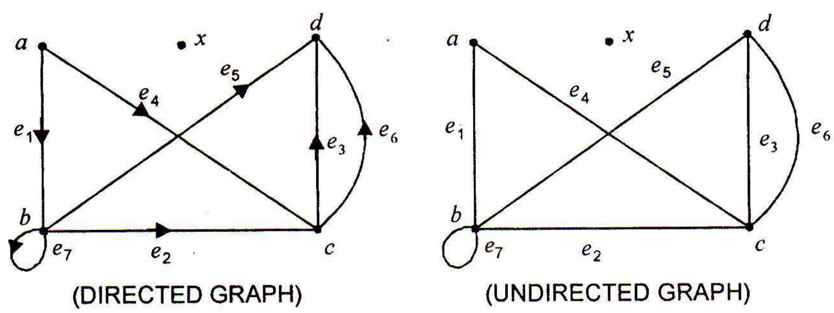



**Fig. 5.1**

**Def. Adjacent Vertices :** Two vertices u and v of a graph G = (V, E) are said to be adjacent if there is an edge e = (u, v) convecting u and v.

**For Example :** In the above diagram, a and b are adjacent vertices since there is an edge $e_1$ = (a, b) joining a and b while the vertices a and d are not adjacent.

**Def. Loop :** An edge that is incident from and into itself is called a loop or self loop or sting.

**For Example :** In the above diagram, $e_1$ = (b, b) is a self loop since it is incident from b onto itself.

**Def. Isolated Vertex :** A vertex of a graph G = (V, E), which is not the end vertex of any edge in G, is called an isolated vertex. In the above diagram, x is an isolated vertex.

**Def. Parallel Edges :** If two (or more) edges of a graph G have the some end vertices, then these edges are called parallel edges. In the above diagram, $e_3$ = (b, d) and $e_6$ = (b, d) are parallel edges.

**Def. Adjacent Edges :** Two non-parallel edges of a graph G are called adjacent if they are incident on a common vertex.

**For Example :** The edges $e_1$ = (a, b) and $e_4$ = (a, c) are adjacent as they have a common end vertex 'a'.

**Note :** It may be noticed that the joining of two vertices in a graph G may be a straight line or a curve, longer or shorter.

## III. Types of Graphs

There are various types of Graphs, which are discussed below :

**I. Simple Graph :** It is a graph which has neither self loop nor parallel edge, as shown below.

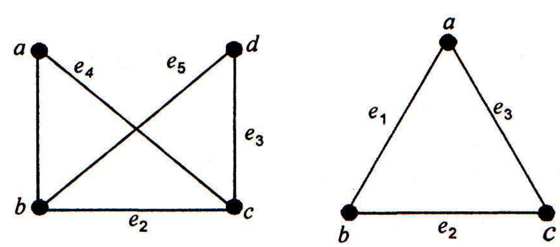



**Fig. 5.2**

**II. General Graph (or Multi Graph) :** It is a graph which has either self loop or parallel edge or both, as shown below

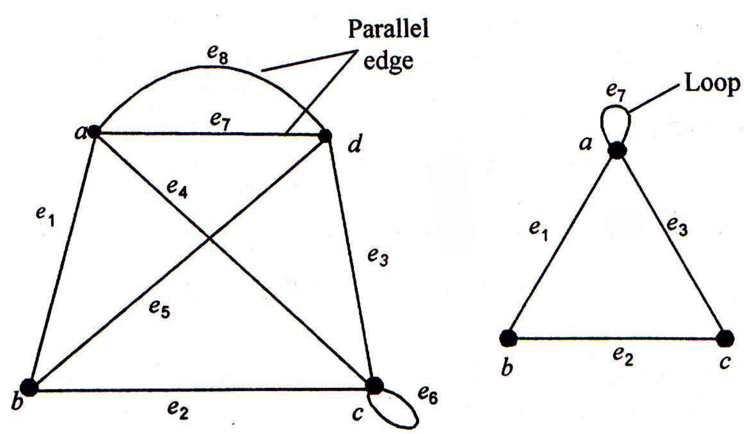



**Fig. 5.3**

**III. Complete Graph :** It may be defined as a simple graph in which there exists an edge between every pair of vertices. It is also called universal graph and a complete graph with n vertices is usually denoted by $k_n$.

**For Example :**

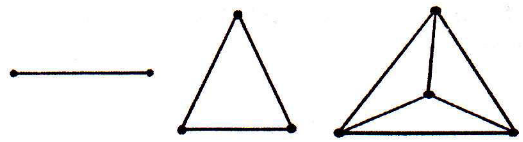


**Fig. 5.4**

**IV. Weighted Graph :** Let G = (V, E) be any graph and w : E → R be a function from edge set E to the set of real numbers R. Then, the graph G = (V,E,w) in which each edge is assigned a number called the weight of the edge, is known as weighted graph.

**For Example :**

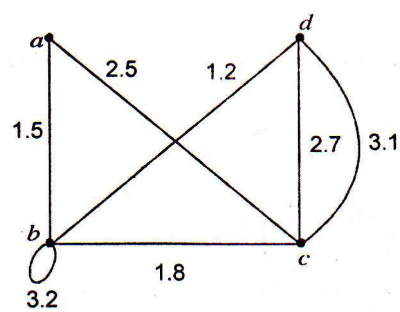
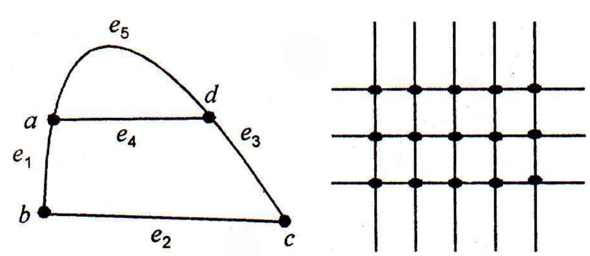



**Fig. 5.5**

**V. Finite and Infinite Graphs :** A graph G = (V,E) is called a finite graph if its vertex set V is a finite set otherwise if its vertex set V is an infinite set, it is called an infinite graph.

**For Example :**




(Finite graph) (Infinite graph)

**Fig. 5.6**

**Note :**

(i) The no. of vertices denoted by $|V(G)|$ is called order of graph G.

(ii) A graph with one vertex and no edges is called trivial graph.

(iii) A graph with no vertices and no edges is called a null or empty graph.

**VI. Regular Graph :** In order to define a regular graph, we first discuss about the degree of a graph G as :

**Def. In-degree :** In a directed graph G, the in-degree of a vertex 'a' is defined as the number of edges which have 'a' as the terminal vertex. It is denoted by $\deg G^+(a)$ or $d^+(a)$.

**Def. Out-degree :** In a directed graph G, the out-degre of a vertex 'a' is defined as the number of edges which have 'a' as the initial vertex. It is denoted by $\deg G^-(a)$ or $d^-(a)$.

**Def. Degree :** The degree of a vertex 'a' in a directed or undirected graph is defined as the total number of edges incident with a. It is denoted by deg G(a) or d(a). Therefore in a directed graph, $\deg G(a) = \deg G^+(a) + \deg G^-(a)$.

**Remark :** A loop contributes two to the degree of a vertexm, since that vertex serves as both end points of the loop.

**For Example :** In the following directed or undirected graph, we have

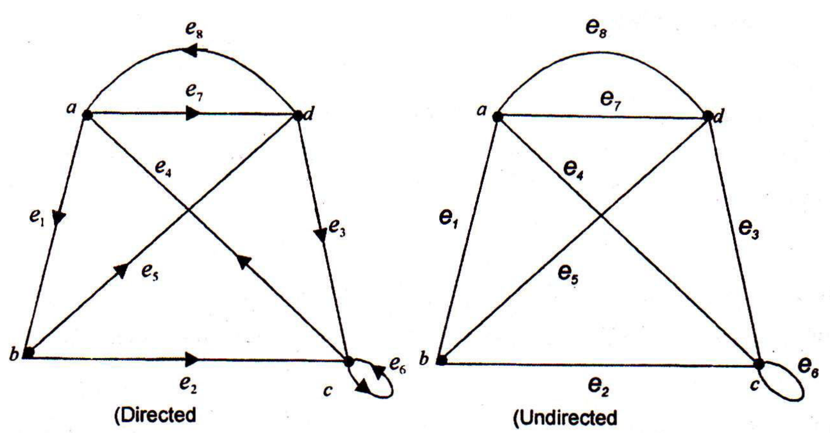



**Fig. 5.7**

| In directed graph | In undirected graph |
|---|---|
| $\deg G(x) = \deg G^+(x) + \deg G^-(x)$ | |
| $\therefore$ $\deg G(a) = 2 + 2 = 4$ | $\deg G(a) = 4$ |
| $\deg G(b) = 1 + 2 = 3$ | $\deg G(b) = 3$ |
| $\deg G(c) = 2 + 3 = 5$ | $\deg G(c) = 5$ |
| $\deg G(d) = 3 + 1 = 4$ | $\deg G(d) = 4$ |

Now, we can define a regular graph as :

A graph in which all the vertices are of same degree, is called a regular graph.

Moreover, if all the vertices have same degree equal to k, then if is called a k-regular graph.

**For Example :**

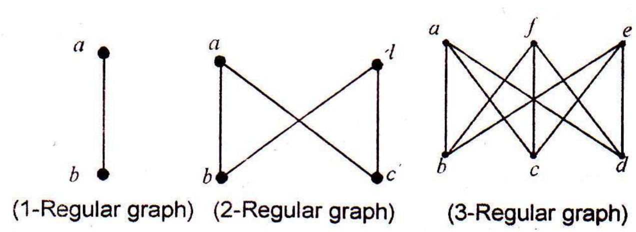



**Fig. 5.8**

**Note :**

(i) A vertex in a directed graph with in-degree zero is called a source and out-degree zero is called a sink.

(ii) The direction of a loop in a directed graph has no significance.

(iii) In a graph G, the vertex v is said to be of even or odd parity according as deg. (v) is even or odd.

(iv) A vertex whose degree in a graph is one, is called pendent vertex.

(v) A vertex whose degree is zero, is d called an isolated vertex.

(vi) A complete graph $K_n$ with n vertices is n-1 regular graph.

**For Example :**

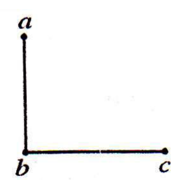



**Fig. 5.9**

Here a and c are pendent vertices, while d is a isolated vertex.

## IV. Graphs Isomorphism and Sub-Graphs

Let G = (V, E) and G' = (V', E') be two graphs. Then, G is isomorphic to G' written as $G \cong G'$ if there exists a bijection f, from v onto v' such that $(v_i, v_j) \in E$, iff $(f(v_i), f(v_j)) \in E'$. We can say, two graphs are isomorphic if there exists a one-one correspondence between their vertices and edges such that incidence relationship is preserved.

**For Example :**

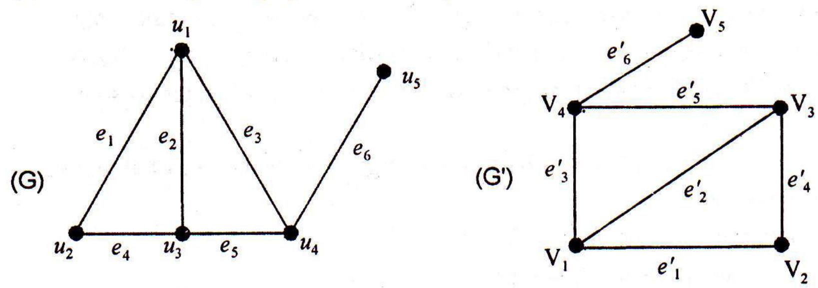



**Fig. 5.10**

Here, $G \cong G'$ since $\exists$ a mapping $u_i \xrightarrow{f} v_i$ and $e_i \xrightarrow{f} e_i^1$ for i = 1, 2, 3, 4, 5.

**Remarks :**

(i) Two isomorphic graphs must have : (a) same number of vertices, (b) same number of edges, and (c) an equal number of vertices with given degrees.

(ii) The converse of remark (i) is not true i.e. the two graphs may be non-isomorphic even though they have the same number of vertices and edges and an equal number of vertices of given degrees.

**For Example :**

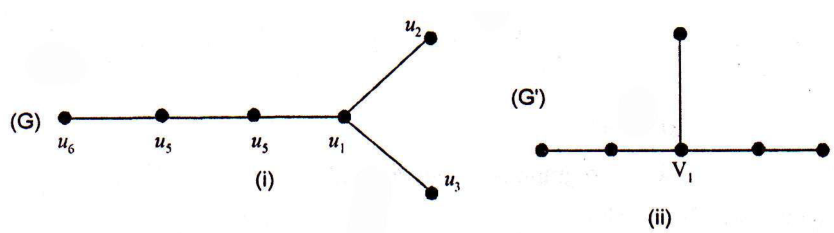



**Fig. 5.11**

Here $G \ncong G'$ since in G' there is only one pendent vertex $v_2$ adjacent to $v_1$, while in G there are two pendent vertices $u_2$ and $u_3$ adjacent to $u_1$.

**Def. Sub-Graph :** Let G and H be two graphs with vertex sets V(H), V(G) and edge sets E(H) and E(G) respecitvely such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$, then H is known as subgraph of G (or G as supergraph of H).

If $V(H) \subset V(G)$, $E(H) \subset E(G)$, then H is a proper subgraph of G and if V(H) = V(G) and $E(H) \subset E(G)$, then H is called a spanning subgraph of G. In simple words, H is a subgraph of G if all the vertices and all the edges of H are in G, and each edge of H has the same end vertices in H as in G.

**For Example :** In the following example, H is a subgraph of G.

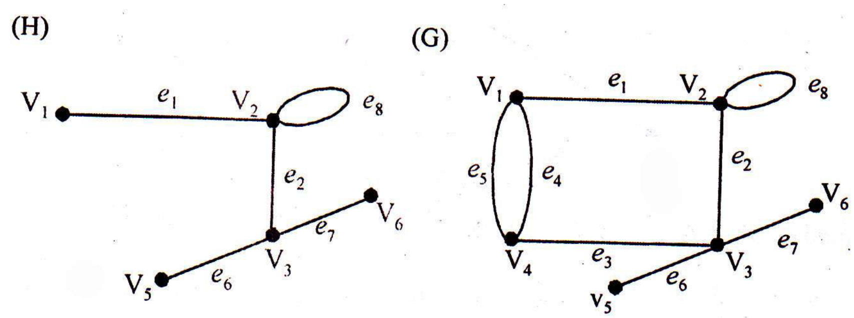



**Fig. 5.12**

**Dif. G–v :** G–v is a subgraph of G obtained by deleting the vertex v from vertex set V(G) and deleting all the edges from edge set E(G) which are incident on v.

**For Example :** Let G be the graph find G–A, G–B, G–C

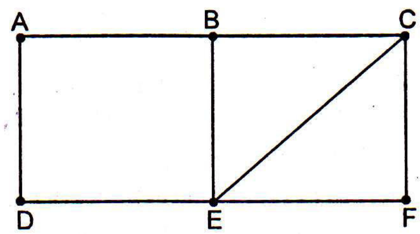



**Fig. 5.13**

Now, G–A, G-B and G–C may be represented as

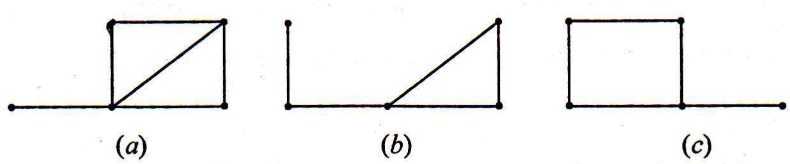



**Fig. 5.14**

**Def. G–e :** Let e be an edge in G. Then, G–e is the subgraph of G obtained by deleting the edge e from the edge set of G.

**For Example :** Let G be a graph.

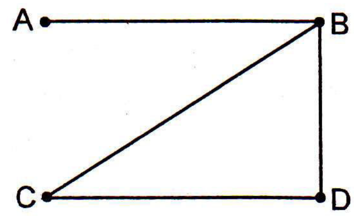



**Fig. 5.15**

Now, following subgrahs may be obtained from it :

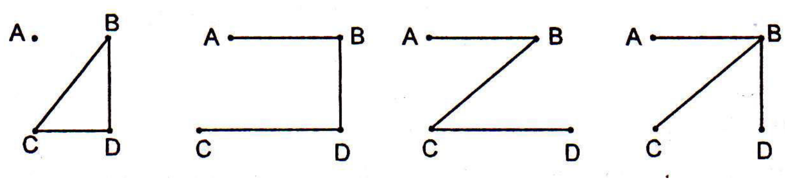



**Fig. 5.16**

**Note :** Every graph is its own subgraph.

**B.A. PART-III** | **MATHEMATICS : PAPER-III (OPT.I)**
**SEMESTER-V** | **DISCRETE MATHEMATICS-I**

---

**LESSON NO. 2.2** | **Author : Dr. Chanchal**

---

# GRAPH THEORY–II

**I. Operations of Graphs**
**II. Some Important Theorems**
**III. Some Important Examples**
**IV. Matrix Representation of Graphs**
**IV. Self Check Exercise**

## I. Operations of Graphs

The following operations may be defined on graphs :

**I. Union of two graphs :** Let $G_1 = (V(G_1), E(G_1))$ and $G_2 = (V(G_2), E(G_2))$ be two graphs. Then, their union denoted by $G_1 \cup G_2$ is a graph $G_1 \cup G_2 = (V(G_1 \cup G_2), E(G_1 \cup G_2))$ such that $V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 \cup G_2) = E(G) \cup E(G_2)$.

**For example :**

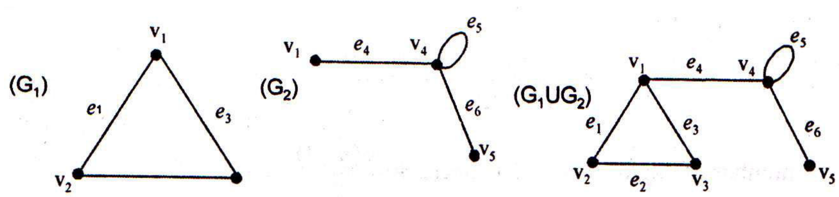



**Fig. 5.17**

**II. Intersection of two graphs :** Let $G_1 = (V(G_1), E(G_1))$ and $G_2 = (V(G_2), E(G_2))$ be two graphs. Then, their intersection denoted by $G_1 \cap G_2$ is a graph $G_1 \cap G_2 = (V(G_1 \cap G_2), E(G_1 \cap G_2))$ such that $V(G_1 \cap G_2) = V(G_1) \cap V(G_2)$ and $E(G_1 \cap G_2) = E(G_1) \cap E(G_2)$.

**For Example :**

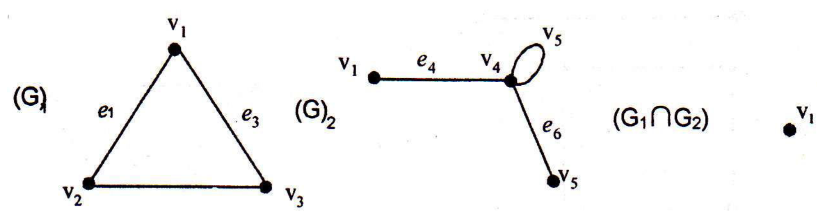



**Fig. 5.18**

**III. Complement of a Graph :** Let G be any graph. Then complement of G denoted by $\bar{G}$ may be defined as the simple graph with the vertex set same as the vertex set of G together with the edge set satisfying the property that there is an edge between two vertices in $\bar{G}$, when there is no edge between these vertices in G.

**For Example :**

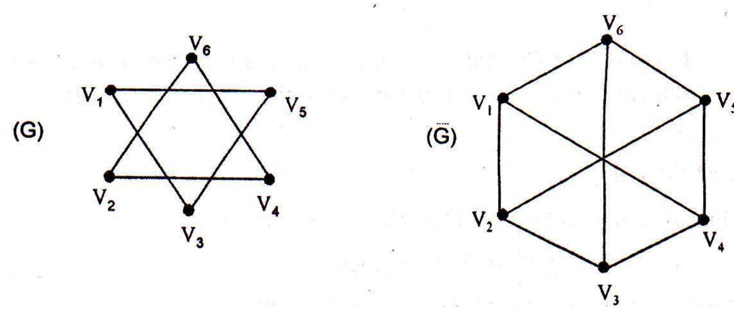



**Fig. 5.19**

**Note :** If the degree of a vertex v in a simple graph G having n vertices is k. Then, degree of v in $\bar{G}$ is n - k - 1.

## II. Some Important Theorems

**Theorem 1 :** (First Theorem on Graph Theory or Handshaking Theorem) : The sum of degrees of all the vertices in a graph G is equal to twice the number of edges in G.

**Proof :** Let 'e' be the number of edges in G and 'n' be the number of vertices in G. Let 'k' be any edge between two vertices $v_1$ and $v_2$ in G. Now, on counting the degree of all vertices in G, k will be counted twice, once in degree of $v_1$ and again in degree of $v_2$. Also, if $v_1$ and $v_2$ are identical, k will be again counted twice since it is a self loop. Hence, every edge is counted twice and total degree is twice the number of edges i.e.

$$\sum_{i=1}^{n} \deg(v_i) = 2e$$

**Theorem 2 :** Prove that in a graph G, the number of vertices of odd degree in even.

**Proof :** Let $v_1$, $v_2$ ............. $v_n$ be n-vertices and $e_1$, $e_2$, .......... $e_e$ be e-edges in G. Then, by first theorem on graph theory

$$\sum_{i=1}^{n} \deg(v_i) = 2e \qquad \text{........ (1)}$$

Now, divide the sum on L.H.S. of (1) in two parts such that one part contains the sum of degree of vertices with even degree and other part contains the sum of degree of vertices with odd degree. Therefore, equation (1) may be written as

$$\sum_{\text{even}} \deg(v_i) + \sum_{\text{odd}} \deg.(v_i) = 2e \quad \ldots\ldots.. (2)$$

Since the R.H.S. of (2) is an even number. Also $\sum_{\text{even}} \deg(v_i)$ is also even. Therefore,

$\sum_{\text{odd}} \deg(v_i)$ or sum of degree of vertices with odd degrees, is also even.

Hence, number of vertices having odd degree mustbe even.

**Theorem 3 :** Prove that maximum degree of any vertex in a simple graph having n vertices is n–1.

**Proof :** Since, in a simple graph, there is no parallel edge and no self loop. Therefore, a vertex can be connected to the remaining n-1 vertices by at most (n-1) edges.
Hence, $\max^m$ degree of any vertex in a simple graph having n vertices is n–1.
Sly, the reader may easily prove that the degree of any vertex in a complete graph having n vertices is n–1.

**Theorem 4 :** Prove that the number of edges in a complete graph with n vertices is $\frac{n(n-1)}{2}$.

**Proof :** Since, degree of any vertex in a complete graph with n vertices is n–1. Now, by first theorem on graph theory we have $\sum_{1=1}^{n} \deg.(v_i) = 2e$

$$\Rightarrow \quad n(n-1) = 2e \qquad [\because \deg.(v_i) = n-1 \text{ for } \le i \le n]$$

$$\Rightarrow \quad e = \frac{n(n-1)}{2}$$

**Theorem 5 :** Show that $\max^m$ no. of edges in a simple graph with n vertices is $\frac{n(n-1)}{2}$.

**Proof :** Do Yourself.

## III. Some Important Examples

**Example 1 :** Prove that there does not exist a graph with 5 vertices with degrees equal to 1, 3, 4, 43 respectively.

**Sol. :** Here n = 5, Let e be the number of edges in graph. Now, by first theorem on graph theory

$$\sum_{i=1}^{n} d(v_i) = 2e \Rightarrow 1 + 3 + 4 + 2 + 3 = 2e$$

$$\Rightarrow \qquad e = \frac{13}{2}, \text{ which is not possible.}$$

Hence, there does not exist a graph with 5 vertices of given degrees.

**Example 2 :** Is there a simple graph G with six vertices of degree 1, 1, 3, 4, 6, 7 ?

**Sol. :** Here n = 6 and we know that, $max^m$ degree of any vertex in a simple graph with n i.e. 6 vertices is n–1 = 6–1 = 5. But G has a vertex of degree 7, which is not possible in a simple graph.

Hence, there is no simple graph G having six vertices with given degrees.

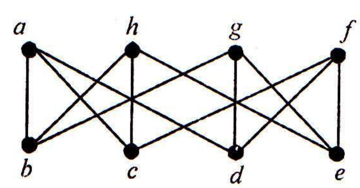
**Example 3 :** Find k, if a k-regular graph with 8 vertices have 12 edges. Also, draw k-regular graph.

**Sol. :** In a k-regular graph, degree of all the vertices is same and equal to k. Here n = 8 and e = 12.

$\therefore$ By first theorem on graph theory $\Rightarrow \sum_{i=1}^{n} d(v_i) = 2e$

$$\Rightarrow \sum_{i=1}^{8} k = 2(12) \Rightarrow 8k = 24 \Rightarrow k = 3$$

Now, the 3-regular graph is




(3-Regular graph)

**Fig. 5.20**

## IV. Matrix Representation of Graphs :

A graph can be represented by a matrix in two ways :

(i) Adjacency matrix (ii) Incidence matrix

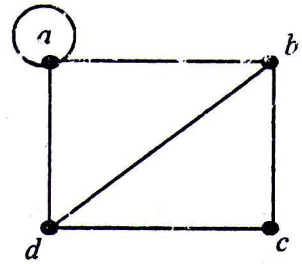
### Adjacency Matrix (for undirected graph) :

Let G be an undirected graph with n vertices. Further suppose G has no multiple edges. Then G is represented by n × n matrix defined as $M = [a_{ij}]_{n \times n}$

$$a_{ij} = \begin{cases} 1 & \text{if } a_i \text{ and } a_j \text{ are adjacent} \\ 0 & \text{otherwise} \end{cases}$$

i.e. an entry is 1 if there is an edge between $a_i$ and $a_j$.

**For Example :** For the graph




The Adjacency matrix is given by

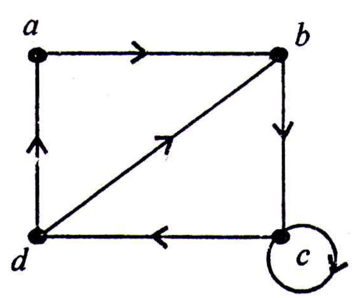
$$\begin{array}{c|cccc} & a & b & c & d \\ a & 1 & 1 & 0 & 1 \\ b & 1 & 0 & 1 & 1 \\ c & 0 & 1 & 0 & 1 \\ d & 1 & 1 & 1 & 0 \end{array} \text{ and so, } M = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

**Note :** Adjacency matrix of undirected graph is always symmetric.

**Adjacency matrix of Directed Graph :** Let G be diagraph with n vertices having no multiple edges. Then G can be represented by n × n adjacency matrix m defined by

$$a_{ij} = \begin{cases} 1 & \text{if there is edge from } a_i \text{ to } a_j \\ 0 & \text{otherwise} \end{cases}$$

**For Example :** The adjacency matrix of following graph is

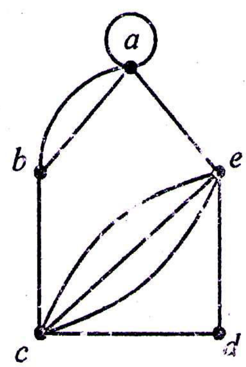


$$\begin{array}{c|cccc} & a & b & c & d \\ \hline a & 0 & 1 & 0 & 0 \\ b & 0 & 0 & 1 & 0 \\ c & 0 & 0 & 1 & 1 \\ d & 1 & 1 & 0 & 0 \end{array} \text{ and so, } M = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

**Adjacency matrix of multi-graph (undirected) :** Let G be undirected graph of n vertices that may contain parallel edges. Then adjacency matrix M is n × n matrix defined by $M = [a_{ij}]_{n\times n}$

$$\text{where } a_{ij} = \begin{cases} n, & n \text{ is number of edges between } a_i \text{ and } a_j \\ 0 & \text{otherwise} \end{cases}$$

**For Example :** The adjacency matrix of Multi-graph.




**is**

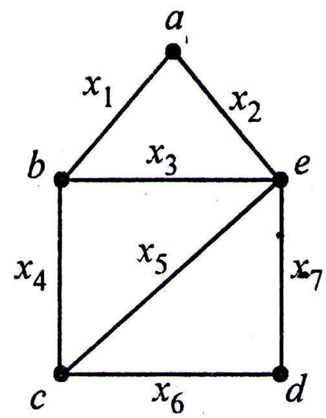
$$\begin{array}{c|ccccc} & a & b & c & d & e \\ \hline a & 1 & 2 & 0 & 0 & 1 \\ b & 2 & 0 & 1 & 0 & 0 \\ c & 0 & 1 & 0 & 1 & 3 \\ d & 0 & 0 & 1 & 0 & 1 \\ e & 1 & 0 & 3 & 1 & 0 \end{array} \text{ and so } M = \begin{bmatrix} 1 & 2 & 0 & 0 & 1 \\ 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 3 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 3 & 1 & 0 \end{bmatrix}$$

**Note :** In similar way, we can find adjacency matrix of directed multi-graph.

**Incidence matrix :** Let G be a graph have m vertices and n edges. Then incidence matrix of graph is m × n matrix written as $A(G) = [a_{ij}]_{n\times n}$ defined by

$$a_{ij} = \begin{cases} 1 & \text{if fth edge } e_i \text{ is incident on ith vertex } v_i \\ 0 & \text{otherwise} \end{cases}$$

**For Example :** For the graph




Number of vertices = 5
Number of edges = 7
So **incidence** matrix is 5 × 7 matrix.

| | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | $x_7$ |
|---|---|---|---|---|---|---|---|
| a | 1 | 1 | 0 | 0 | 0 | 0 | 0 |
| b | 1 | 0 | 1 | 1 | 0 | 0 | 0 |
| c | 0 | 0 | 0 | 1 | 1 | 1 | 0 |
| d | 0 | 0 | 0 | 0 | 0 | 1 | 1 |
| e | 0 | 1 | 1 | 0 | 1 | 0 | 1 |

so $$A(G) = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

## V. Self Check Exercise

1. How many nodes or vertices are necessary to construct a 2-regular graph with exactly 6 edges.

2. Find n, if a complete graph with n vertices has 15 edges.
3. Does there exist a 3-regular graph with nine vertices ?
4. Is it possible to construct a graph with 12 edges such that two of its vertices have degree 3 and remaining vertices have degree 4.

**Suggested Readings :**

1. Dr. Babu Ram, Discrete Mathematics
2. C.L. Liu, Elements of Discrete Mathematics (Second Edition), McGraw Hill, International Edition, Computer Science Series, 1986.
3. Discrete Mathematics, S. Series.
4. Kenneth H. Rosen, Discrete Mathematics and its Applications, McGraw Hill Fifth Ed. 2003.

**LESSON NO. 2.3** **Author : Dr. Chanchal**

# GRAPH THEORY–III & TREES AND FINITE STATE MACHINES

## Structure :



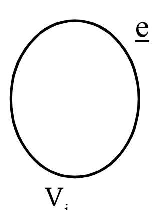
## I. Walk, Path and Circuit

A path II in a grah G consists of a pair of (V$\Pi$ E$\Pi$) of sequences:- a vertex sequence V$\Pi$ : $V_1, V_2, \ldots\ldots\ldots\ldots, V_{k-1}$, and an edge sequence E$\Pi$ $e_1, e_2, \ldots\ldots\ldots\ldots, e_{k-1}$, for which (i) Each success pair $V_i$, $V_i$ + 1 of uertices is adjacet i G, and edge ei has $V_i$, $V_i$ + 1 as end point for i=1,2--- k-1.

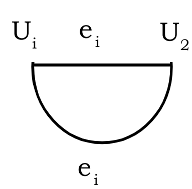
A Circuit is a path that begins and ends at the same vertex. It is also as cycle of circular path al polygon

A path is called simple if no vertex appears more than once in the vertex sequence, except possibly if $V_i$, $U_k$ (In this case, the Path is called a simple circuit).

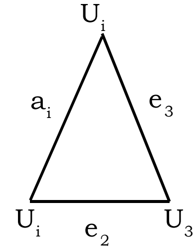
K-cycle:- A cycle with k-edges is called a k-cycle and it is denoted by $C_k$




Self Loop
1-cycle
$W=v_1 e\ v_1$




2-cycle
$K_i = e_1 v_2 e_2 v_1$
Pair of parallel Edge




3-cycle
Triangle
$v_1 e_1 v_2 e_2 u_3 e_3 U_1$

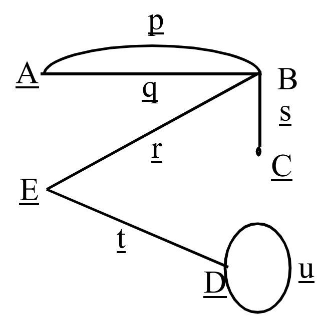
**Note:- n-cycle is a polygon of n sides.**

**Def. Length of Path :** The number of edges appearing in the sequence of the path is called the length of the path.

**Remark.** (i) An edge which is not a self loop is a path of length 1.

(ii) A self loop can be included in a walk but not in a path.

(iii) The terminus vertices of a path are of degree 1 and the internal vertices of the walk are of degree 2.




$V_{\Pi_1}$ : A, B, C, D, D,

$E_{\Pi_I}$ : p,r,t,u

A,p,B,r,e,t,D,u,D | Path but not simple

$V_{\Pi_2}$ : A, B, A

$E_{\Pi_I}$ : p,q

$E_{\Pi_I}$ **:** D,E,B,C | Simple Path

$E_{\Pi_3}$ **:** t,r,s

**Defn:-** A graph is called Connected if there is a path from any vertex to any other vertex in the graph. Otherwise, the graph is disconnected. If the graph is disconected, the various connected pieces are called the components if the graph.

e.g:- The graph in above Fig. (A) is conected while the graph in Fig. B is disconected. Moreover, it has two connected components.

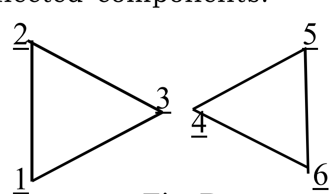



Fig. B

Defn:-Discrete Graph:- A graph denoted by, U$\underline{n}$ ($n \geq 1$), with n vertices and no edge is called a Discrete Graph on n vertices.

e.g. . .   . . . . .

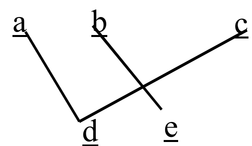
$U_2$   $U_5$

Defn Linear Graph:- A graph, deneted by L $\underline{n}$($n \geq 1$), with vertices ($u_{1,} u_1 + 1$) for $I < I < n$, is called linear Graph on n verticesl

e.g.    

$L_2$   $L_4$

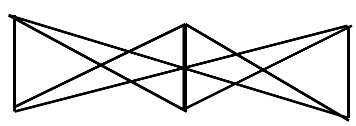
**Bipartise Graph:-** Let G by any graph. if vertex set V ca be partitioned into two disjoint subsets A and B $\ni$ Biery edge in G joins a vertex in A and a vertex in B, the the graph is said to be Bipartise graph.




**Complete Bipartise Graph:-** A BG is said to be complete if every vertex A is joined to every vertex is B.

Km, n m- No. of vertex in A

n- No. of vertex in B



$K_{3,3}$

## II. Planar Graphs

A planar graph is a graph drawn in the plane in such a way that no two edges intersect (cross) each other.

**Def. Planar graph :** A planar graph is a graph which is isomorphic to a plane graph i.e., it can be redrawn as a plane graph.

A graph which is not a planar graph is called **non-planar graph.**

**For Example :** (i) The complete graph with four vertices $K_4$ is usually drawn with crossing edge see fig (a). But it can also be drawn with non-crossing edges see fig. (b)

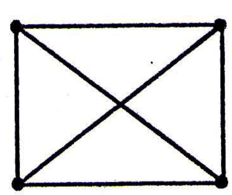


Fig. (a)

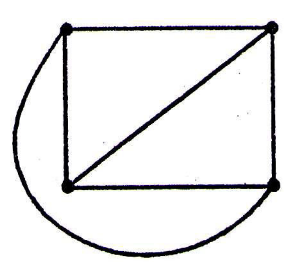


Fig. (b)

Hence $K_4$, is a planar graph.

(ii) A complete graph is five vertices is non-planar i.e., $K_5$ is non-planar.

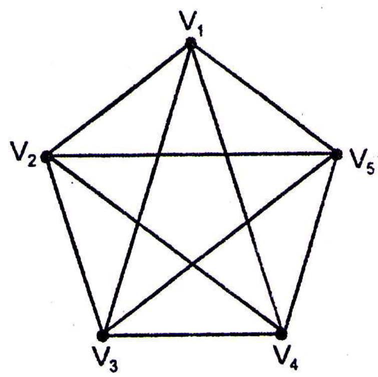



Fig (*a*)

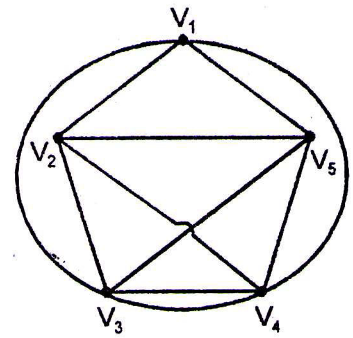



Fig (*b*)

Since the graph shown in fig. (a) cannot be drawn in plane without crossing edges see fig. (b). Hence $K_5$ is non-planar graph.

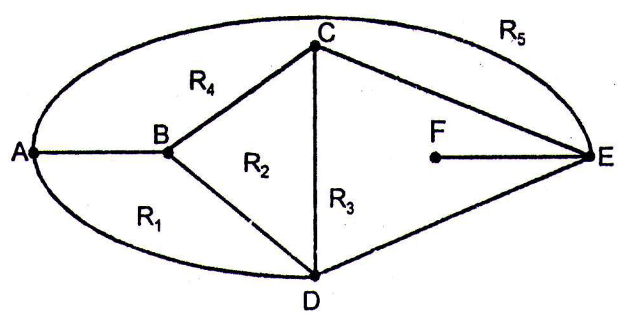
**Region :** A plane graph partitions the plane into several regions. These regions are called faces. Each region is depicted by the set of edges.

**Cycle :** The boundary of the region R of graph G is cycle if the boundary of R contains no cut edges of G. i.e., contain no edge such that on removing any edge in R it will not be a closed circuit.

**Degree of face :** If G be graph and g be its face, then the number of edges in the boundary of g with cut edges counting twice is defined as the degree of face g.

**Cut Edge :** Cut edge in a graph is an edge whose removal results in a disconnected graph.

**For Example :** Consider the following plane graph




Various regions are shown by $R_1$, $R_2$, $R_3$, $R_4$, $R_5$

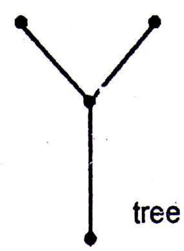
Here deg ($R_1$) = 3, deg ($R_2$) = 3, deg ($R_3$) = 5, deg ($R_4$) = 4, deg ($R_5$) = 3

## Theorem 1 Euler's Formula

Let G = (V, E) be a connected planar graph and let R be the number of regions defined by any planar depiction of G. Then $R = |E| - |V| + 2$

**Proof :** We prove the result by induction let k be the number of regions determined by G.

We first show that the result is true for k = 1. A tree determine the above regon, for example




No. of vertices = 4, No of edges = 3. Also from the formula, we have

$$1 = |E| - |V| + 2 \Rightarrow |E| = |V| - 1$$

i.e., No. of edges = No. of vertices −1, which is always true for a tree.

$\therefore$ The result is true for k = 1.

Let us assume that the result is true for all $k \geq 1$. Let G be a connected plane graph determining (k + 1) regions. Remove an edge which is common to the boundary of two regions. We obtain a graph G' having k-regions.

Let $|V'|$, $|E'|$, R' denote respectively the number of vertices number of edges and regions of G', then

$$R' = |E'| - |V'| + 2 \qquad \ldots (1)$$

Also, we have

$$|V'| = |V|,\ |E'| = |E| - 1,\ R' = R-1$$

$$\therefore \quad |E| - |V| + 2 = |E'| + 1 - |V'| + 2 = (|E'| - |V'| + 2) + 1 = R' + 1 \ [\because \text{of } (1)]$$
$$= R$$

$\therefore$ results is true for k + 1

$\therefore$ result follows by induction for all connected graphs.

## III. Shortest Path Problem

Let G be a connected graph whose edges are assigned unique weights (taken as distances). We want to determine shortest possible path between a pair of vertices. Method for this was developed by Dijkstra and is known as Dijkstra's algorithm.

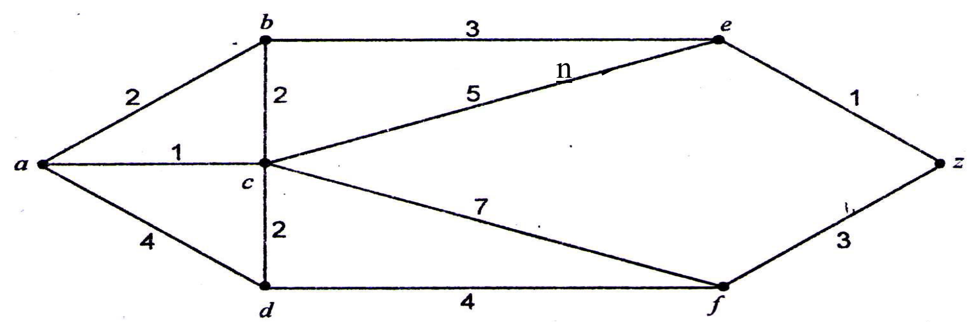
## Dijkstra's Algorithm :

This algorithm maintains a sets of vertices whose shortest path from source is already known. If there is no path from source vertex do any other vertex then it is represented by + ∞. All weights must be positive.

Following points are considered :

1. Initially there is no vertex in sets.
2. Include source vertex $V_s$ in S. Determine all the paths from $V_s$ to al other vertices without going through any other vertex.
3. Include that vertex in S which is nearest to $V_s$ find shortest paths to all the vertices through this vertex, give the values.
4. Repeat the process until (n–1) vertices are not included in S.

**Example :** Find the shortest path between a and z.




**Step I :** Include the vertex a in S and determine all the direct paths from a to all other vertices without going through any other vertices.

**Distance to all other vertices**

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| a | a | b | c | d | e | f | z |
| | 0 | 2(a) | 1(a) | 4(a) | ∞ | ∞ | ∞ |

**Step II :** Include vertex in S, nearer to a and determine shortest path to all the vertices through this vertex. The nearest vertex is c.

**Distance to all other vertices**

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| a, c, | a | b | c | d | e | f | z |
| | 0 | 2(a), | 1(a) | 3(a,c) | 6(a,c) | 8(a,c) | ∞ |

**Step III :** Second nearest vertex is b

**Distance to all other vertices**

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| a, c, b | a | b | c | d | e | f | z |
| | 0 | 2(a) | 1(a) | 3(a,c) | 5(a,b) | 8(a,c) | ∞ |

**Step IV :** Next vertex is d.

**Distance to all other vertices**

| | a | b | c | d | e | f | z |
|---|---|---|---|---|---|---|---|
| a, c, b,d | a | b | c | d | e | f | z |
| | 0 | 2(a) | 1(a) | 3(a,c) | 5(a,b) | 7(a,c) | $\infty$ |

**Step V :** Next vertex is e.

**Distance to all other vertices**

| | a | b | c | d | e | f | z |
|---|---|---|---|---|---|---|---|
| a, c, b,d,e | a | b | c | d | e | f | z |
| | 0 | 2(a) | 1(a) | 3(a,c) | 5(a,b) | 7(a,c) | 6(a,b,e) |

**Step VI :** Next vertex is z.

**Distance to all other vertices**

| | a | b | c | d | e | f | z |
|---|---|---|---|---|---|---|---|
| a, c, b,d,e,z | a | b | c | d | e | f | z |
| | 0 | 2(a) | 1(a) | 3(a,c) | 5(a,b) | 7(a,c) | 6(a,b,e) |

n – 1 vertices are included in S.

So minimum path between a and z is 6

Path is $a \rightarrow b \rightarrow e \rightarrow z$.

## IV. Euler Paths and Circuits

**Euler Path :** A simple path in a graph G is called Euler Path if it traverses every edge of graph exactly once.

**Euler Circuit :** Euler Circuit is a circuit in graph G which traverses every edge of graph exactly once. Euler Circuit is simply a closed Euler path. It is also called Euler line.

**Eulerian Graph :** A graph which contain either Euler Path or Euler Circuit is called Eulerian Graph.

**For Example :**

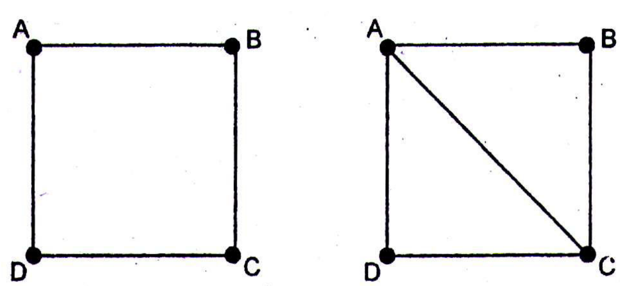



Fig. I Fig. II

Fig. I has Euler Circuit A B C D A

Fig. II has Euler Path A B C D A C

**Remark :** Circuit starts and ends at same vertex whereas path starts and ends at different vertices.

**Some Important Results :**

**Result 1 :** A connected graph G is a Euler Graph if all the vertices of G are of even degree.

**Result 2 :** If G is a connected graph and every vertex of G has even degree, then prove that G has a Euler Circuit.

**Result 3 :** If a graph has Euler Path then it has either no vertex of odd degree or two vertices of odd degree.

## V. Hamiltonian Paths and Circuits

**Hamiltonian Path :** A Hamiltonian Path in connected graph is a path which contains each vertex of graph exactly once.

**Hamiltonian Circuit :** A Hamiltonian circuit is a circuit that contains each vertex of graph exactly once except for the first vertex, which is also the last.

**Hamiltonain Graph :** A graph which possesses either Hamiltonian circuit or Hamiltonian path is called a Hamiltonian graph.

**Remarks :** I In Hamiltonian circuit or path we have to visit all the vertices. There may be some unvisited edges.

II. If G has n vertices, then Hamiltonian circuit will contain n edges where as Hamiltonian Path will contain n – 1 edges.

III. There may be more than one Hamiltonian paths and circuits in a graph.

**Some Important Results :**

**Result I :** Let G be a connected simple graph with n vertices, n > 2. Let U and V are any two non-adjacent vertices in G and deg (U) + deg (V) $\geq$ n, then G is Hamiltonian.

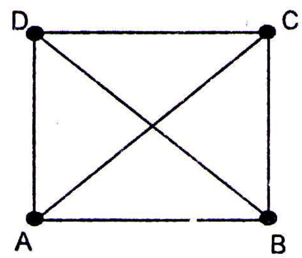
**Result II :** Let G be a connected simple graph with n vertices, n > 2. If deg(V) $\geq \frac{n}{2}$ for every V $\in$ G then G is Hamiltonian.

**Result III :** Let m be the number of edges in Graph G. If m $\geq \frac{1}{2}(n^2 - 3n + 2)$ where n is number of vertices of G then G is Hamiltonian.

**Hamiltonion Circuit in Complete Graph :**

Let $K_n$ be complete graph of n vertices, n $\geq$ 3. Then $K_n$ will definitely contain a Hamiltonian circuit. In fact $K_n$ will contain $\frac{\lfloor n-1}{2}$ Hamiltonian circuits.

**For Example :** Consider the graph $K_4$.




Then by above result, $K_4$ contains $\frac{\underline{|4-1}}{2} = 3$. Hamiltonian circuits which are ABCDA, ABDCA and ADBCA.

## VI. Some Important Examples

**Example 1 :** Give an example of a connected graph that has

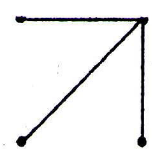
(a) Neither an Euler circuit nor a Hamiltonian circuit.

(b) An Euler circuit but not Hamiltonian circuit.

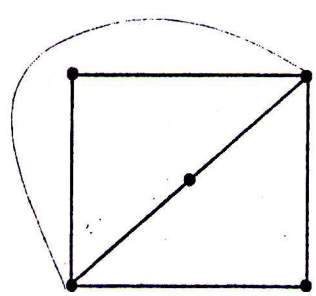
(c) A Hamiltonian circuit, no Euler circuit.

(d) Both an Hamiltonian circuit and Euler circuit.

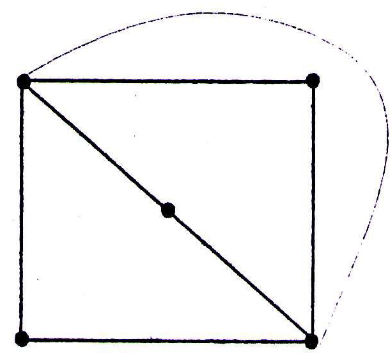
**Sol.** (a) A connected graph that has neither an Euler circuit nor a Hamiltonian circuit is



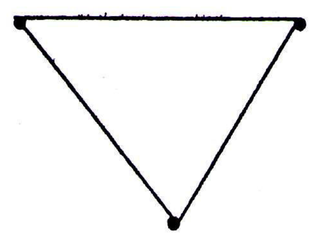
(b) A connected graph that has an Euler circuit but not Hamiltonian circuit is



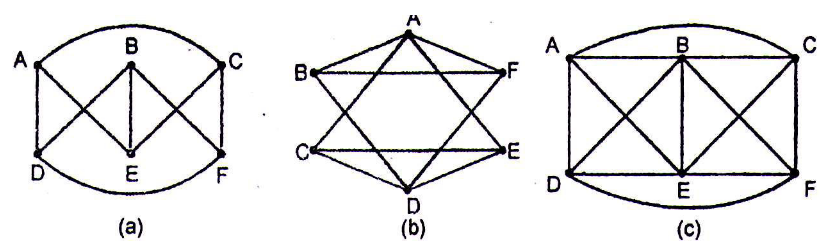
(c) A connected graph that has a Hamiltonian circuit and no Euler circuit is



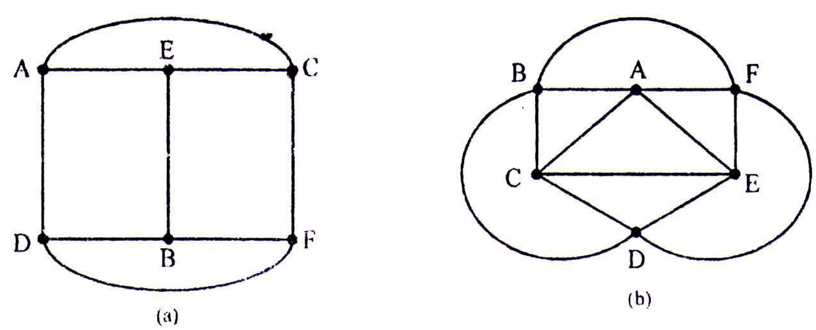
(d) A connected graph that has both is Hamiltonian circuit and Euler circuit is



**Example 2 :** Draw a planar representation of each of the following graphs.




**Sol. :** The planar representation of the graph (a) and (b) are




The graph (c) is non-planar.

**Example 3 :** Determine the number of regions defined by a connected planar graph with 4 nodes and 8 edges. Draw such a graph.

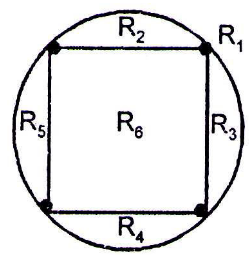
**Sol.** Here |V| = 4, |E| = 89

∴ By Euler's formula.

$$R = |E| - |V| + 2 = 8 - 4 + 2 = 6$$

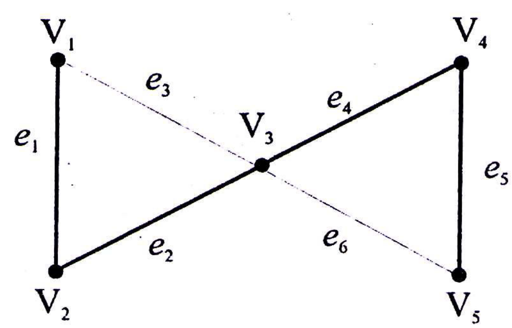
∴ The given connected graph has 6 regions.

The required graph is.




**Example 4 :** Find Hamiltonian paths for each of the following graphs and show that no Hamiltonian circuit exits.

**Sol.** The Hamiltonian path for the graph (a) is as shown by the heavy lines.

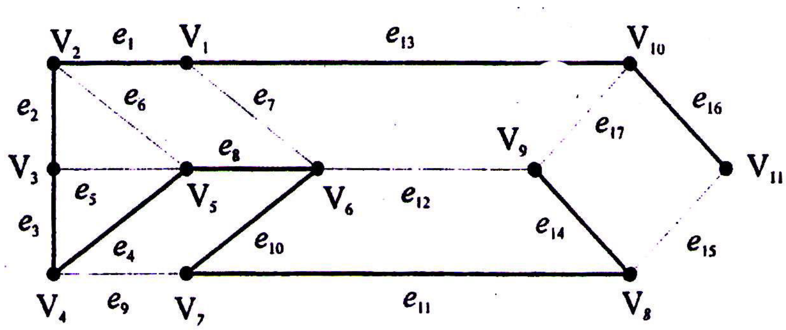



i.e. $W = v_1 e_1 v_2 e_2 v_3 e_4 v_4 e_5 v_5$

is a Hamiltonian path

clearly the graph has no Hamiltonian circuit (as it has to cross the vertex $V_3$ twice)

The Hamiltonian path for the graph (b) is as shown by the heavy line




i.e. W = $v_{11}\ e_{16}\ v_{10}\ e_{13}\ v_1\ e_1\ v_2\ e_2\ v_3\ e_3\ v_4 e_4\ v_5 e_8\ v_6\ e_{10}\ v_7\ e_{11}\ v_8\ e_{14}\ v_9$

is a Hamiltonian path

clearly the graph has no Hamiltonian circuit (as it has to cross the vertex $V_8$ or $V_9$ or $V_{10}$ or $V_{11}$ twice).

## VII. Travelling Salesman Problem

Suppose a salesman wants to visit a certain number of cities starting from his headequarters. The distances (or cost or time) of journey between every pair of cities, denoted by $c_{ij}$, that is, distance from city i to city j is assumed to be known. The problem is:

Salesman starting from his home city visited each city only once and returns to his home city in the shortest possible distance (or at the least cost or in the least time).

Given n cities and distance $c_{ij}$, the salesman starts from city 1, then any permutation of 2, 3,...., n represents the number of possible ways for his tour. So, there are (n–1)! possible ways for his tour. The problem is to select an optimal route that could achieve his objective.

The problem may be classified as :

*(i)* *Symmetrical :* If the distance between every pair of cities is independent of the direction of his journey.

*(ii)* *Asymmetrical :* For one or more pair of cities the distance changes with the direction.

## Example 5

A machine operator processes five types of items on his machine each week, and must choose a sequence for them. The set-up cost per change depends on the item presently on the machine and the set-up to be made according to the following table.

| From item | To item | | | | |
|---|---|---|---|---|---|
| | A | B | C | D | E |
| A | ∞ | 4 | 7 | 3 | 4 |
| B | 4 | ∞ | 6 | 3 | 4 |
| C | 7 | 6 | ∞ | 7 | 5 |
| D | 3 | 3 | 7 | ∞ | 7 |
| E | 4 | 4 | 5 | 7 | ∞ |

If he processes each type of item once and only once each week, how should be sequence the items on his machine in order to minimise the total set-up cost?

**Solution :**

Step 1: Reduce the cost matrix using Step 1 and 2 of the Hungarian algorithm and then make assignments in rows and columns having single zeros as usual.

| | | | | |
|---|---|---|---|---|
| ∞ | 1 | 3 | $\boxed{0}$ | 1 |
| 1 | ∞ | 2 | $\cancel{0}$ | 1 |
| 2 | 1 | ∞ | 2 | $\boxed{0}$ |
| $\cancel{0}$ | $\boxed{0}$ | 3 | ∞ | 4 |
| $\boxed{0}$ | $\cancel{0}$ | $\cancel{0}$ | 3 | ∞ |

Step 2: Note that row 2 is not assigned. So, mark √ to row 2. Since there is a zero in the 4th column of the marked row, we tick 4th column. Further, there is an assignment in the first row of 4th column. So, tick first row. Draw lines through all unmarked rows and marked columns. We can find the number of lines is 4 which is less than the order of the matrix. So, go to next step (see table).

| | | | | | |
|---|---|---|---|---|---|
| ∞ | 1 | 3 | $\boxed{0}$ | 1 | √ |
| 1 | ∞ | 2 | $\cancel{0}$ | 1 | √ |
| 2 | 1 | ∞ | 2 | $\boxed{0}$ | |
| $\cancel{0}$ | $\boxed{0}$ | 3 | ∞ | 4 | |
| $\boxed{0}$ | $\cancel{0}$ | $\cancel{0}$ | 3 | ∞ | |
| | | | √ | | |

Step 3: Subtract the lowest element from all the elements not covered by these lines and add the same with the elements at the intersection of two lines. Then we get the table as:

| | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| 1 | $\infty$ | $\cancel{0}$ | 2 | $\boxed{0}$ | $\cancel{0}$ |
| 2 | $\boxed{0}$ | $\infty$ | 1 | $\cancel{0}$ | $\cancel{0}$ |
| 3 | 2 | 1 | $\infty$ | 3 | $\boxed{0}$ |
| 4 | 0 | $\boxed{0}$ | 3 | $\infty$ | 4 |
| 5 | $\cancel{0}$ | $\cancel{0}$ | $\cancel{0}$ | 4 | $\infty$ |

The optimum assignment is $1\to4$, $2\to1$, $3\to5$, $4\to2$, $5\to3$ with minimum cost as Rs. 20.

This assignment schedule does not provide us the solution of the travelling salesman problem as it gives $1\to4$, $4\to2$, $2\to1$, without passing through 3 and 5.

Next, we try to find the next best solution which satisfies this restriction. The next minimum (non-zero) element in the cost matrix is 1. So, we bring 1 into the solution. But the element '1' occurs at two places. We consider all cases separately until we get an optimal solution.

We start with making an assignment at (2, 3) instead of zero assignment at (2,1). The resulting assignment schedule is

$$1 \to 4,\ 4 \to 2,\ 2 \to 3,\ 3 \to 5,\ 5 \to 1$$

When an assignment is made at (3, 2) instead of zero assginment at (3, 5), the resulting assignment schedule is

$$1 \to 5,\ 5 \to 3,\ 3 \to 2,\ 2 \to 4,\ 4 \to 1$$

The total set-up cost in both the cases is 21.

## Example 6

Solve the travelling salesman problem given by the following data :

$$c_{12} = 20,\ c_{13} = 4,\ c_{14} = 10,\ c_{23} = 5,\ c_{34} = 6$$
$$c_{25} = 10,\ c_{35} = 6,\ c_{45} = 20 \text{ where } c_{ij} = c_{ji}$$

and there is no route between cities i and j if the value for cij is not shown.

**Solution :** The cost matrix is :

| | | | | |
|---|---|---|---|---|
| ∞ | 20 | 4 | 10 | ∞ |
| 20 | ∞ | 5 | ∞ | 10 |
| 4 | 5 | ∞ | 6 | 6 |
| 10 | ∞ | 6 | ∞ | 20 |
| ∞ | 10 | 6 | 20 | ∞ |

Repeating the steps as before using the Hungarian algorithm, the optimum table obtained is :

| | | | | |
|---|---|---|---|---|
| ∞ | 12 | $\boxed{0}$ | $\cancel{0}$ | ∞ |
| 11 | ∞ | $\cancel{0}$ | ∞ | $\boxed{0}$ |
| $\cancel{0}$ | 1 | ∞ | $\cancel{0}$ | 1 |
| $\boxed{0}$ | ∞ | $\cancel{0}$ | ∞ | 9 |
| ∞ | $\boxed{0}$ | $\cancel{0}$ | $\boxed{8}$ | ∞ |

The solution is

$$1 \rightarrow 3,\ 3 \rightarrow 4,\ 4 \rightarrow 1,\ 5 \rightarrow 2,\ 2 \rightarrow 5$$

which is not the solution of the travelling salesman problem as the sequence obtained is not in the cyclic order.

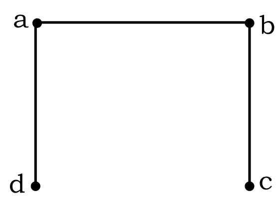
The next lowest number (other than 0) is 1. Therefore, make an assignment in the cell (3, 2) having the element 1. Consequently, make an assignment in the cell (5, 4) having element 8, instead of zero element in the cell (5, 2). The assignment Table is

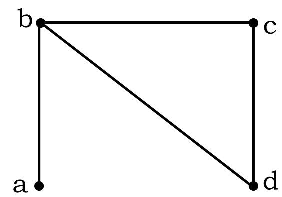
| | | | | |
|---|---|---|---|---|
| ∞ | 12 | $\boxed{0}$ | $\cancel{0}$ | ∞ |
| 11 | ∞ | $\cancel{0}$ | ∞ | $\boxed{0}$ |
| $\cancel{0}$ | $\boxed{1}$ | ∞ | $\cancel{0}$ | 1 |
| $\boxed{0}$ | ∞ | $\cancel{0}$ | ∞ | 9 |
| ∞ | $\cancel{0}$ | $\cancel{0}$ | $\boxed{8}$ | ∞ |

The shortest path for the travelling salesman is 1>3, 3>2, 2>5, 5>4, 4>1

## VIII. Introduction to Trees :

A graph G is called a tree if it is connected and contains no cycles.

For example : 

is a tree but 

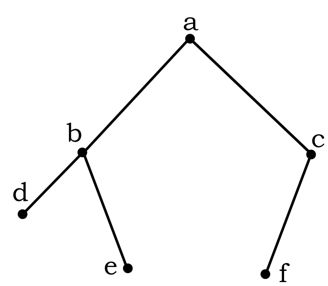

is not a tree as it contains a cycle b,c,d,b.

**Note :** (1) A tree has to be a simple graph i.e. having neither a self loop nor parallel edges because both of them form cycles.

(2) A tree is said to be directed if every edge of tree is assigned a direction, otherwise it is undirected.

**Some Basic Terms Used in Trees :**

Consider the following tree




**(i)** **Node :** It is the key component of tree which stores information and can have one or more links for connecting to other nodes.

**(ii)** **Edge :** A directed line from one node to another mode is called edge, link, arc or branch of a tree.

**For example :** ab, ac are edges.

**(iii)** **Root :** The vertex having indegree zero is called root of tree.

**For example :** a is the root of above tree.

**(iv)** **Path :** A path is a sequence of nodes when we traverse from one node to other along the edges which connect them.

**e.g. :** Path from a to f is a, c, f.

**(v)** **Level :** Level of node is an integer value that measures the distance of a node from the root.

**e.g. :** Root is at level O, The Child(s) of root are at level 1 and so on.

**(vi) Depth :** The depth of node is the length of path from node to root of tree. Root has depth O.

**(vii) Rooted Tree :** A rooted tree is a directed tree which contains a unique vertex with in-degree zero and every other vertex has in-degree one.
**e.g. :** The above tree is a rooted tree with root 'a'.

**(viii) Parent and Offspring :** If (x, y) is any directed edge, then x is called parent of y and y is called offspring of x. Root of tree has no parent whereas every other node has a unique parent. A parent can have several offsprings which is also called child or son.
**e.g. :** a is the parent of b and c. b has two offsprings d and e.

**(ix) Leaf :** A node having no offsprings (outdeg ree = 0) is called a leaf or external node or terminal node.
**e.g. :** d, e, f are leaves.

**(x) Siblings :** Two nodes having same parent are called siblings.
**e.g. :** b,c are siblings of a.

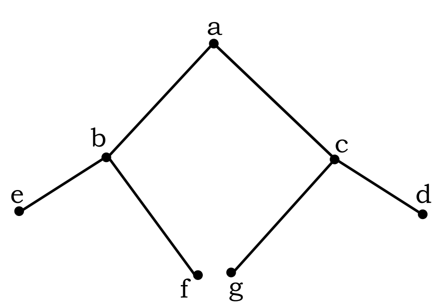
**(xi) Interior Node :** Node with atleast one child.

**(xii) Forest :** It is an undirected graph whose components are all trees.

**Binary Tree :**

A tree T is called n-Tree or n-ary tree if every vertex has atmost n offsprings. In particular, if n = 2, then tree is called binary tree or It is that tree in which every node can have 0, 1, or 2 offsprings. Moreover, if in n-tree, every vertex of T other than leaves, has exactly n-offsprings, then T is called complete n-Tree. For n = 2, it is complete binary tree.

**For example :**



is a binary tree.

Moreover, every vertex (except leaves) has 2 children, so it is complete binary tree.

## IX. Properties of Trees :

Here, we state some important properties of trees (the proof is left as an exercise for the reader) :

**Property I :** There is one and only one path between every pair of vertices in a tree T.

**Property II :** If in a graph G, there is one and only one path between every pair of vertices, then G is tree.

**Property III :** A tree with n vertices has n–1 edges.

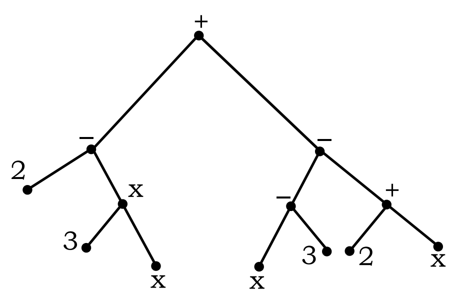
**Property IV :** A graph is a tree iff it is minimally connected.

**Property V :** A graph G with n vertices, (n–1) edges and no circuit is connected.

**Property VI :** In any non-trivial tree, there are atleast two vertices of degree–1 (or two pondent vertices)

**Labelled Trees :** A tree is paid to be labelled in which every vertex of tree has assigned a unique label.

### For Example :




is labelled binary tree for the expression

$(2 - (3 \times x)) + ((x - 3) - (2 + x))$.

## X. Traversal of Binary Trees or Tree Searching

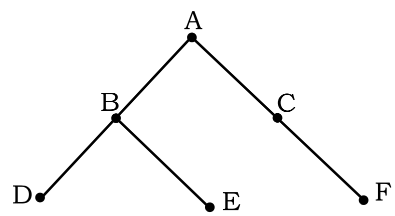
Traversing means to visit each node of tree exactly once. The three standard ways of traversing a binary tree T with root $R_1$ are :

**Preorder :** Process the root R. Then, traverse the left subtree of R in preorder and then traverse the right subtree of R in preorder.

**Inorder :** Traverse the left subtree of R in inorder. Process the root R and then traverse the right subtree of R in inorder.

**Postorder :** Traverse the left subtree of R in post order, then traverse the right subtree of R in postorder and then process the root R.

**For Example :**




For the tree :

Preorder Traversal : A, B, D, E, C, F

Postorder Traversal : D, E, B, F, C, A

Inorder Traversal : D, B, E, A, C, F

## XI. Spanning Tree

Let G be a connected graph. A subgraph T is called spanning tree if

(i) T is true and

(ii) T contains all vertices of G.

**e.g.**


T is a spanning tree for the connected graph G.

**Remarks :**

(1) A spanning tree of graph is not unique.

(2) A graph G is connected iff it has a spanning tree (Proof : Do Yourself)

(3) Cayley's Theorem : The complete graph $K_n$ has $n^{n-2}$ different spanning trees.

**Minimal Spanning Tree :** It is a spanning tree of a weighted graph, with the condition that sum of weights of tree is as small as possible.

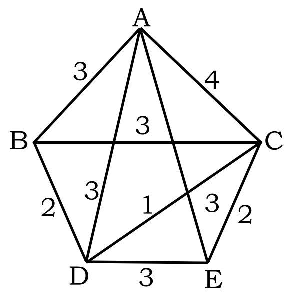
Kruskal's Algorithm (To find minimal spanning tree)

Let G be the given connected graph. Then, the algerithm involves following steps :

1. Write all the edges of graph in increasing order of their weight.
2. Select the smallest edge of G.
3. For each successive step, select another smallest edge of G which makes no cycle with previously selected edges.
4. Go on repeating step 3 until n–1 edges have been selected. The sum of

weights of there n–1 edges will constitute required minimal spanning tree.

**e.g.**

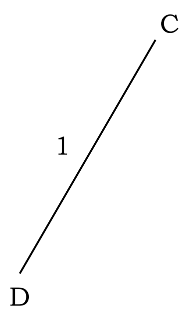



Here, no. of vertices (n) = 5 Firstly, we write all the edges in increasing order of their weight.

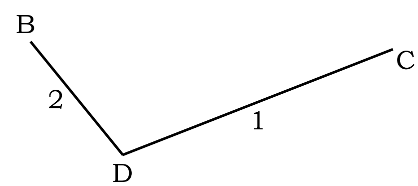
E = {CD, BD, CE, AB, BC, AD, AE, DE, AC}

We start from edge CD and then select edges one by one from E until we select 4 (i.e. n–1) edges.

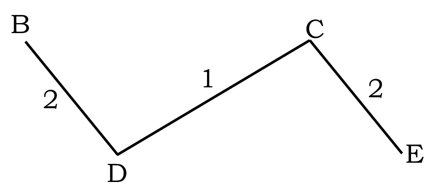
(i)



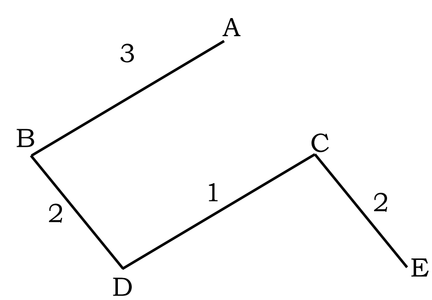

(ii) Select next edge BD




(iii) Select next edge CE




(iv) Select next edge AB



Since, we have selected 4 edges, so we stop here. Sum of weights is
1+2+2+3 = 8

**Self Check Exercise :**

1. Give an example of graph that has
   (i) Euler Circuit but not Hamiltonian Circuit.
   (ii) Hamiltonian Circuit but not Euler Circuit.
2. Let G = (V, E) be a simple, connected Planar graph with more than one edge, then the following inequalities holds.
   (i) $2\,|E| \geq 3R$ (ii) $|E| \leq 3\,|V| - 6$
   (iii) There is a vertex v of G such that $\deg(v) \leq 5$.

**XII. Suggested Readings :**

1. Dr. Babu Ram, Discrete Mathematics
2. C.L. Liu, Elements of Discrete Mathematics (Second Edition), McGraw Hill, International Edition, Computer Science Series, 1986.
3. Discrete Mathematics, S. Series.
4. Kenneth H. Rosen, Discrete Mathematics and its Applications, McGraw Hill Fifth Ed. 2003.