# Centre for Distance and Online Education

# Punjabi University, Patiala

**Class : M.sc. (IT)-I** **Semester : I**
**Paper : MITM11OT (Mathematical Foundation of Computer Science)** **Unit-I**
**Medium : English**

***Lesson No.***



***Department website : www.pbidde.org***

**MITM1104T: Mathematical Foundation of Computer Science**

**Maximum Marks: 70** **Maximum Times: 3 Hrs.**
**Minimum Pass Marks: 35%**

**Course Objective:** The purpose of this course is to provide a clear understanding of the concepts that underlying fundamental concepts and tools in discrete mathematics with emphasis on their applications to computer science. It emphasizes mathematical definitions and proofs as well as applicable method. On completion of this course, the students will be able to

- Be familiar with the basic terminology of functions, relations, and sets and demonstrate knowledge of their associated operations.
- Master to solve advanced mathematical problems, apply various methods of mathematical proof, and communicate solutions in writing
- Master to comprehend advanced mathematics, and present the material orally and in writing
- Utilize the knowledge of computing and mathematics appropriate to the discipline.
- Evaluate mathematical principles and logic design

## Course Content

### SECTION A

Logic: Propositions, Implications, Precedence of logical operators, Translating English sentences, System specifications, Propositional equivalences, Predicates and Quantifiers, Nested quantifiers, Order of quantifiers. Sets, Power set, Set operations, Functions, One-to-One functions and Onto functions, Inverse and composition of functions, Floor function, Ceiling function.
Algorithms, Searching algorithms, Sorting, Growth of functions, Big-O notation, Big-Omega and Big-Theta notation, Complexity of algorithms, Mathematical induction, The Basics of counting, The Pigeonhole principle.

### SECTION B

Recurrence relations, Solving recurrence relations, Divide and Conquer algorithms, Generating functions for solving recurrence relations, Inclusion-Exclusion Principle.
Relations and their properties, n-ary relations and their applications, Representing relations, Closure of relation, Equivalence relations, Partial ordering.
Graphs: Introduction, Terminology, Representing graphs and graph isomorphism, Connectivity, Euler and Hamiltonian paths, Shortest path problems, Planar graphs.

**Pedagogy:**
The Instructor is expected to use leading pedagogical approaches in the class room situation, research-based methodology, innovative instructional methods, extensive use of technology in the class room, online modules of MOOCS, and comprehensive assessment practices to strengthen teaching efforts and improve student learning outcomes.

The Instructor of class will engage in a combination of academic reading, analyzing case studies, preparing the weekly assigned readings and exercises, encouraging in class discussions, and live project based learning.

**Text and Readings:** Students should focus on material presented in lectures. The text should be used to provide further explanation and examples of concepts and techniques discussed in the course:

- Rosen, K.H: Discrete Mathematics and Its Applications, TMH Publications.
- Discrete and Combinational Mathematics, Ralph P. Grimaldi, Pearson Education.
- Elements of Discrete Mathematics, C. L. Luie, TMH Publications.
- Discrete Mathematics, Richard Johnson, Baugh, Pearson Education.
- Discrete Mathematical Structures with Applications to Computer Science, J. P. Tremblay& R. P. Manohar, MGH Publications.
- Discrete Mathematical Structures, B.Kotman, R.C. Busbay, S.Ross, PHI.

**Scheme of Examination**

- English will be the medium of instruction and examination.
- Written Examinations will be conducted at the end of each Semester as per the Academic Calendar notified in advance
- Each course will carry 100 marks of which 30 marks shall be reserved for internal assessment and the remaining 70 marks for written examination to be held at the end of each semester.
- The duration of written examination for each paper shall be three hours.
- The minimum marks for passing the examination for each semester shall be 35% in aggregate as well as a minimum of 35% marks in the semester-end examination in each paper.
- A minimum of 75% of classroom attendance is required in each subject.

**Instructions to the External Paper Setter**

The question paper will consist of three Sections: A, B and C. Sections A and B will have four questions each from the respective section of the syllabus and will carry 10.5 marks for each question. Section C will consist of 7-15 short answer type questions covering the entire syllabus uniformly and will carry a total of 28 marks.

**Instructions for candidates**

- Candidates are required to attempt five questions in all, selecting two questions each from section A and B and compulsory question of section C.
- Use of non-programmable scientific calculator is allowed.

**LESSON NO. 1.1** **Author : Dr. Chanchal**
**LAST UPDATED ON MAY, 2023**

# PROPOSITIONAL CALCULUS

**Structure :**



## 1.1.1 Objectives

The prime objective of this unit is to enlighten the basic concepts of logic, prepositions, quantifiers, set theory and functions. In this lesson, we are going to explain the theory of logic to study that how we can arrive at a conclusion from known statements by using the laws of logic.

## 1.1.2 Introduction

Firstly, we introduce some basic terms associated with prepositional calculus.

**Def. Sentence :** It is sensible combination of words.

**For example :** Sun is a heavely body.

**Def. Statement or Proposition :** A statement is a declarative sentence which is either true or false but not both. The truth or falseness of a statement is called its truth value. In simple words, a statement is a sentence in the grammatical sense conveying a situation which is neither imperative, interrogative nor exclamatory.

**For Example :** (i) "May God bless you with happiness !". This sentence is not a statement because of its exclamation mark.

(ii) "$(x-1)^2 = x^2 = 2x + 1$". This is a statement and its truth value is T or 1. It should be noted that a mathematical identity is always a statement.

Now, a statement or preposition is of two types : simple and compound. Any statement whose truth or otherwise does not explicitly depend on another statement is said to be simple but a compound statement is combination of two or more simple statements. Moreover, the phrases or words which connect two simple statements are called logical connectives or simply connectives and some of these are "and", "or", "not", "if then", "if and only if".

**For Example :** "8 is an even number" is a simple statement while "If you work hard, then you will pass" is a compound statement.

The simple statements which are combined to form compound statements, are called components. Our problem is to determine the truth value of a compound statement from the truth values of their components and for this purpose, we draw truth table consisting of columns and rows. The number of columns depends upon the number of simple statements and relationships among them but number of rows depends only upon the number of simple statements. The truth tables are very helpful in finding out the validity of a report.

### 1.1.3 Logical Statements

There are various types of logical compound statements, which are discussed below :

#### I. Conjunction of Original Statements

Any two statements can be combined by the connective "and" to form compound statement called the "conjunction" of original statements.

**For Example :** The conjunction of "He is practical" and "He is sensitive" is "He is practical and sensitive". In symbols, if two statements are denoted by p, q then, their conjunction is denoted by $p \wedge q$ (read as "p and q").

**Rule :** $p \wedge q$ is true when p and q are true.

**Truth Table for ^**

| p | q | p^q |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | F |
| F | F | F |

**Table 1.1**

**Example :** Let p : 5 + 7 = 12 and q : 2 is a prime number.

$\therefore$ $p \wedge q$ : 5 + 7 = 12 and 2 is a prime number. Now, p and q both are true, therefore p ^ q is true.

**II. Disjunction of Original Statements**

Any two statements can be conbined by the connective "or" to form compound statement called the "Disjunction" of original statements.

**For Example :** The disjunction of "I shall watch the game on television" and "I shall go to college" is "I shall watch the game on television or go to college".

In symbols, the disjunction of two statements p and q is denoted by p v q (read as "p or q")

**Rule :** p v q is false when both p and q are false otherwise it is true.

**Truth Table for V**

| p | q | p v q |
|---|---|---|
| T | T | T |
| T | F | T |
| F | T | T |
| F | F | F |

**Example :** Let p : 5 < 12 and q : 8 + 3 = 12.

∴ p v q : 5 < 12 or 8 + 3 = 12.

Here p is true and q is false. Therefore, p v q is true.

**III. Negation (or Denial) of a Statement**

To every statement, there corresponds a statement which is its negation that refers to contradiction. The best way to write the negation of given statement is to put in the word "not" at the proper place or to put the phrase "It is not the case that" in the beginning. Negation of a statement p is denoted by "~p".

**For Example :** If p : He is a good student. Then, ~p : He is not a good student or It is not the case that he is a good student. We cannot say that "He is a bad student" is the negation of p.

**Rule :** If p is true, then ~p is false and vice versa.

**Truth Table for ~**

| p | ~ p |
|---|---|
| T | F |
| F | T |

**IV. Conditional Statement**

Let p and q be two statements. Any statement of the form "if p then q" is called a conditional statement. It is denoted by p → q (read as p conditional q or p implies q).

Here, p is sufficient for q but not essential i.e. there can be q, even without p.

**For example :** Let p : you work hard and q : you will pass. Now, $p \rightarrow q$ : If you work hard, then you will pass.

**Rule :** $p \rightarrow q$ is true in all cases except when p is true and q is false.

**Truth Table for →**

| p | q | $p \rightarrow q$ |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | T |
| F | F | T |

### V. Biconditional Statement or Equivalence

Let p and q be two statements. Any statement of the form "p if and only if q" is called a biconditional statement, denoted by $p \leftrightarrow q$.

**Rule :** $p \leftrightarrow q$ true if both p and q have the same truth value and false if p and q have apporite truth values.

**Truth Table of ↔**

| p | q | $p \leftrightarrow q$ |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | F |
| F | F | T |

### VI. Converse, Inverse and Contrapositive

If $p \rightarrow q$ is a direct statement, then

(i) $q \rightarrow p$ is called its converse,

(ii) $\sim p \rightarrow \sim q$ is called its inverse

and (iii) $\sim q \rightarrow \sim p$ is called its contrapositive

**Note :** Since $p \rightarrow q = \sim q \rightarrow \sim q$, $\therefore$ contrapositive $\equiv$ direct statement

and $q \rightarrow p = \sim p \rightarrow \sim q$, $\therefore$ converse $\equiv$ inverse

**Exercise :** Write truth tables for $q \rightarrow p$, $\sim p \rightarrow \sim q$ and $\sim q \rightarrow \sim p$.

### VII. Dual of a Statement

As we know that dual relationship between 'line' and 'point' exists through the interchange of the words 'meet' and 'join'.

**For Example :** Dual of "A line is the join of two points" is "A point is the meet of two lines". Similarly, to find the dual of any statement in logic, we first interchange v and ^.

**For Example :** Dual of ~ (p v q) = (~p) ^ (~q) is ~(p^q) = (~p) v (~q).

### VIII. Tautologies and Contradictions (or Fallacies)

A tautology is a proposition which is true for all the truth values of its components and a contradiction is a porposition which is false for all the truth values of its components.

**For Example :** p v ~p is a tautology and p ^ ~p is a contradiction, as shown below.

**Truth Table**

| p | ~p | p v ~p | p ^ ~p |
|---|---|---|---|
| T | F | T | F |
| F | T | T | F |

**Art 1.1 :** Prove De-Morgan's laws using conjuction and disjunction.

Or

Prove that (i) ~ (p ^ q) = ~p v ~q

(ii) ~ (p v q) = ~p ^ ~q

**Proof :** **Truth Table**

(i)

| p | q | ~ p | ~ q | p ^ q | ~ (p ^ q) | ~ p v ~ q |
|---|---|---|---|---|---|---|
| T | T | F | F | T | F | F |
| T | F | F | T | F | T | T |
| F | T | T | F | F | T | T |
| F | F | T | T | F | T | T |

As the last two columns of truth table are same, therefore ~ (p ^ q) = ~p v ~q.

(ii) Do Yourself.

**Art 1.2 :** Prove that : (i) p → q = (~p) v q

(ii) ~ (p → q) = p ^ ~q

**Proof :** (i)

| p | q | ~p | p → q | (~p) v q |
|---|---|---|---|---|
| T | T | F | T | T |
| T | F | F | F | F |
| F | T | T | T | T |
| F | F | T | T | T |

Since the last two columns are same, therefore $p \rightarrow q = (\sim p) \vee q$

(ii) Do Yourself.

**Example 1 :** Write down the truth table for the statement $(p \rightarrow q) \leftrightarrow (\sim p \vee q)$

**Sol.**

| p | q | $p \rightarrow q$ | ~p | ~pvq | $(p \rightarrow q) \leftrightarrow (\sim p \vee q)$ |
|---|---|---|---|---|---|
| T | T | T | F | T | T |
| T | F | F | F | F | T |
| F | T | T | T | T | T |
| F | F | T | T | T | T |

**Example 2 :** (i) If p stands for the statement 'I do not like chocolates' and q for the statement 'I like ice-cream', then what does ~p^q stand for ?

(ii) If p stands for the statement, 'I will not go to school' and q for the statement, 'I will watch a movie', then what does ~p v q stand for ?

**Sol. :** (i) p : I do not like chocolates, q : I like ice-cream, ~p^q: I like chocolates and ice-cream.

(ii) p : I will not go to school, q : I will watch a movie, ~p v q : Either I will go to school or I will watch a movie.

**Example 3 :** Write the following statement in symbolic form and give its negation : If it rains, he will not go to school.

**Sol. :** Let p : It rains, q : He will go to school

∴ symbolic expression is $p \rightarrow \sim q$

Its negation is $\sim(p \rightarrow \sim q) = \sim(\sim p \vee \sim q) = \sim(\sim p) \wedge \sim(\sim q) = p \wedge q$

In words : Even if it rains, he will go to school.

**Example 4 :** Prove that if $p \rightarrow q$ and $q \rightarrow r$, then $p \rightarrow r$.

**Sol. :** Here we are given that $p \rightarrow q$ , $q \rightarrow r$ and we have to prove that $p \rightarrow r$. The result will be established if we show that $[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$ is a tautology.

**Truth Table**

| p | q | r | $p \rightarrow q$ | $q \rightarrow r$ | $p \rightarrow r$ | $(p \rightarrow q) \wedge (q \rightarrow r)$ | $[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$ |
|---|---|---|---|---|---|---|---|
| T | T | T | T | T | T | T | T |
| T | F | T | F | T | T | F | T |
| F | T | T | T | T | T | T | T |
| F | F | T | T | T | T | T | T |
| T | T | F | T | F | F | F | T |
| T | F | F | F | T | F | F | T |
| F | T | F | T | F | T | F | T |
| F | F | F | T | T | T | T | T |

So, if p→q and q→r, then p→r.

**Example 5 :** State the converse and contrapositive of the implication "If it snows tonight, then I will stay at home."

**Sol. :** Let p : It snows tonight, q : I will stay at home. Converse of statement p→q is q→p i.e., "If I stay at home then it snows tonight."

Contrapositive of statement p→q is ~q→~p i.e., "If I do not stay at home then it will not snow tonight".

**1.1.4 Validity of Arguments**

Firstly, we define an argument as :

**Def. Argument :** An argument is a statement which assets that given set of propositons $p_1, p_2, p_3, \ldots\ldots, p_n$ taken together gives another proposition P. These are expressed as $p_1, p_2, p_3, \ldots\ldots p_n /\!-P$. The sign "/–" is spoken as turnstile. The porpositions $p_1, p_2, p_3, \ldots\ldots p_n$ are called "premises" or "assumptions" and P is called the "conclusion".

**Valid Argument :-** An argument $p_1, p_2, p_3, \ldots\ldots p_n /\!-P$ is true whenever all the premises $p_1, p_2, p_3, \ldots\ldots p_n$ are true, otherwise the argument is false. A true argument is called valid argument and a false argument is called a fallacy. The validity can also be judged by the relationship $p_1 \wedge p_2 \wedge p_3 \ldots\ldots \wedge p_n \rightarrow P$ provided it is a tautology.

**Example 6 :** Test the validity of :

Unless we control population, all advances resulting from planning will be nullified. But this must not be allowed to happen. Therefore we must somehow control population.

**Sol. :** Let the symbols for the statements be :

p : we control the population

q : all advances resulting from planning are nullified.

∴ the argument is $\sim p \rightarrow q, \sim q /\!-p$

**Truth Table**

| p | q | ~ p | ~ q | ~ p → q | (~ p → q) ^ ~ q | [(~p → q) ^ ~ q] → p |
|---|---|---|---|---|---|---|
| T | T | F | F | T | F | T |
| T | F | F | T | T | T | T |
| F | T | T | F | T | F | T |
| F | F | T | T | F | F | T |

Since [(~p → q) ^ ~ q) → is tautology.

∴ the given argument is valid.

**Example 7 :** Check the validity of argument :

If I work, I cannot study. Either I work or pass mathematics.

I passed mathematics. Therefore, I study.

**Sol. :** Let p : I work, q : I sudy, r : I pass mathematics

The given statement is [(p → ~ q) ^ (p V r) ^ (r)] → q

**Truth Table**

| p | q | r | ~q | p → ~q | p V r | I<br>(p→ ~q) ^ (p V r) ^ (r) | II<br>1→q |
|---|---|---|---|---|---|---|---|
| T | T | T | F | F | T | F | T |
| T | T | F | F | F | T | F | T |
| T | F | T | T | T | T | T | F |
| T | F | F | T | T | T | F | T |
| F | T | T | F | T | T | T | T |
| F | T | F | F | T | F | F | T |
| F | F | T | T | T | T | T | F |
| F | F | F | T | T | F | F | T |

The given statement is not a tautology.

So, argument is not valid

### 1.1.5 Proposition Generated by a Set

Let S be any set of propositions. A proposition generated by S is any valid combination of propositions in S with conjunction, disjunction and negation.

**Note :** The conditional and biconditional operators are not included as they can be obtained from conjunction, disjunction and negation.

**Equivalence**

Let S be a set of propositions and p, q be propositions generated by S. p and q are equivalent if p ↔ q is a tautology. The equivalence of p and q is denoted by p ↔ q.

**Implication**

Let S be a set of propositions and p, q be propositions generated by S. p implies q if p → q is a tautology. p ⇒ q is written to indicate the implication.

**Laws of Logic**

Here 0 stands for cortradiction, 1 for tautology.

**Commutative Laws**

p V q ⇔ q V p          p ^ q ⇔ q ^ p

**Associative Laws**

(p V q) v r ⇔ p V (q V r)          (p ^ q) ^ r ⇔ p ^ (q ^ r)

**Distributive Laws**

$p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$ $p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$

**Identity Laws**

$p \wedge 0 \Leftrightarrow p$ $p \vee 1 \Leftrightarrow p$

**Negation Laws**

$p \wedge \sim p \Leftrightarrow 0$ $p \vee \sim p \Leftrightarrow 1$

**Idempotent Laws**

$p \vee p \Leftrightarrow p$ $p \vee p \Leftrightarrow p$

**Null Laws**

$p \wedge 0 \Leftrightarrow 0$ $p \vee 1 \Leftrightarrow 1$

**Absorbtion Laws**

$p \wedge (p \vee q) \Leftrightarrow p$ $p \vee (p \wedge q) \Leftrightarrow p$

**DeMorgan's Laws**

$\sim (p \vee q) \Leftrightarrow (\sim p) \wedge (\sim q)$ $\sim (p \wedge q) \Leftrightarrow (\sim p) \vee (\sim q)$

**Involution Laws**

$\sim (\sim p) \Leftrightarrow p$

**Common Implication and Equivalence**

**Detachment**

$(p \rightarrow q) \wedge p \Rightarrow q$

**Contrapositive**

$(p \rightarrow q) \wedge \sim q \Rightarrow \sim p$

**Disjunctive Additon**

$p \Rightarrow (p \vee q)$

**Conjunctive Simplification**

$(p \wedge q) \Rightarrow p$ and $(p \wedge q) \Rightarrow q$

**Disjunctive Simplification**

$(p \vee q) \wedge \sim p \Rightarrow q$ and $(p \vee q) \wedge \sim q \Rightarrow p$

**Chain Rule**

$(p \rightarrow q) \wedge (q \rightarrow r) \Rightarrow p \rightarrow r$

**Conditional Equivalences**

$(p \rightarrow q) \Leftrightarrow (\sim q \rightarrow \sim p) \Leftrightarrow (\sim p \vee q)$

**Biconditional Equivalences**

$(p \leftrightarrow q) \Leftrightarrow ((p \rightarrow q) \wedge (q \rightarrow q)) \Leftrightarrow ((p \wedge q) \vee (\sim p \wedge \sim q))$.

**Note :** All the above laws, implications and equivalences can be proved very easily with the help of truth tables.

### 1.1.6 Proposition Over a Universe

Let U be a non-empty set. A proposition over U is a sentence that contains a variable that can take on any value in U and which has a definite truth value as a result of any such substitution.

**For Example :** Consider $7x^2 - 6x = 0 \Rightarrow x(7x - 6) = 0$

$\Rightarrow \quad x = 0, \frac{7}{6}$

If we take **Q** as universe, then truth set (i.e., solution set) of $7x^2 - 6x = 0$ is $\left\{0, \frac{7}{6}\right\}$.

If we take **Z** as universe, then truth set of $7x^2 - 6x = 0$ is {0}.

**Truth Set** If p (n) is a proposition over U, then the truth set of p (n) is

$T_{p(n)} = \{a \in U / p(a) \text{ is true}\}$

Consider the set {1, 2, 3, 4}

Its power set is {$\phi$, {1}, {2}, {3}, {4}, {1, 2}, {1, 3}, {1, 4}, {2, 3}, {2, 4},{3, 4}, {1, 2, 3}, {1, 2, 4}, {1, 3, 4}, {2, 3, 4}, {1, 2, 3, 4}}

Let proposition be $\{1, 2\} \cap A = \phi$

$\therefore$ truth set of proposition taken over the power set of {1, 2, 3, 4} is {$\phi$, {3}, {4}, {3, 4}}.

**Tautology and contradiction :** A proposition over U is a tautology if its truth set is U. It is a contradiction if its truth set is empty.

**Equivalence :** Two propositions are equivalent if $p \leftrightarrow q$ is a tautology. In other words, p and q are equivalent if $T_p = T_q$.

**Example :** x + 7 = 12 and x = 5 are equivalent propositions over the integers.

**Implication :** If p and q are propositions over U, then p implies q if $p \rightarrow q$ is a tautology. In other words $p \rightarrow q$ when $T_p \subseteq T_q$.

**Example :** Over the natural numbers,

$n \leq 3 \rightarrow n \leq 8$ as $\{0, 1, 2, 3\} \subseteq \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$

**Truth Set of Compound Propositions**

The truth sets of compound propositions can be expressed in terms of the truth sets of simple propositions. The following list gives the connection between compound and simple truth sets :

1. $T_{p \wedge q} = T_p \cap T_q$
2. $T_{p \vee q} = T_p \cup T_q$
3. $T_{\sim p} = T_p^{\,c}$
4. $T_{p \leftrightarrow q} = \left(T_p \cap T_q\right) \cup \left(T_p^{\,c} \cap T_q^{\,c}\right)$
5. $T_{p \rightarrow q} = T_p^{\,c} \cup T_q$.

### 1.1.7 Predicates

A predicate may be defined as a declarative sentence whose truth/false value depends on one or more variables. When the variables are given specific values, the sentence becomes a statement. Moreover, we use the function notation to denote predicates. For example : P(x) = "x is even", and $Q(x_1y)$ = "x is heavier than y" are predicates. Out of these, P(8) is true, while the statement Q (feather, brick) is false. Moreover, some domain can be defined for each predicate. For example, domain for P(x) could be the integers while for $Q(x_1y)$, domain could be some collection of physical objects.

### 1.1.8 Quantifiers

If p(n) is a propositions over U with $T_{p(n)} \neq \phi$, then we say "There exists an n in U such that p (n) is true." We abbreviate this sentence as $(\exists\ n)_t$, (p(n)). This is known as **Essential quantifier**.

It is clear that if p (n) is a proposition over a universe U, its truth set $T_{p\ (n)}$ is a subset of U.

**For Examples :**

(i) $(\exists\ k)_z$, (5k = 100) means that there is an integer k such that 100 is a multiple of 5. This is true.

(ii) $(\exists\ x)_Q$, $(x^2 - 3 = 0)$ means that there is a rational number x such that $x^2 = 3$. This is false as the solution set of the equation $x^2 - 3 = 0$ over **Q** is empty. We write it as

$((\nexists\ x)_Q, (x^2 - 3 = 0)$

If p (n) is a proposition over U, with $T_{p(n)} = U$. Them we say "for all n in U, p(n) is true." We abbreviate this as $(\forall\ n)_t$, (p (n)). $\forall$ is known as universal quantifier.

## Negation of Quantified Proposition

When we negate a quantified proposition, then the universal and existential quantifiers become complement of one another. In simple words, negation of an existentially quantified proposition is a universally quantified proposition and negation of a universally quantified proposition is an existentially quantified proposition. In symbols,

$$\sim (\forall\ n)_U\ (p\ (n)) \Leftrightarrow (\exists\ n)_U\ (\sim p\ (n))$$

and $$\sim (\exists\ n)_U\ (p(n)) \Leftrightarrow (\forall\ n)_U\ (\sim p\ (n)).$$

**Example 8 :** Over the universe of positive integers : p(n) : n is prime and n < 32, q (n) : n is power of 3, r (n) : n is a divisor of 27.

(a) What are the truth sets of these propositions ?

(b) Which of the three propositions implies one of the others ?

**Sol.** We have

(a) $T_p = \{2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31\}$

$T_q = \{1, 3, 6, 9, 12, 15, 18, 21, \ldots\ldots\ldots\}$

$T_r = \{1, 3, 9, 27\}$

(b) Since $T_r \subseteq T_q$

$\therefore$ r implies q.

**Example 9 :** Translate in your own words and indicate whether it is true or false that : $(\exists x)_Q (3x^2 - 12 = 0)$

**Sol. :** Consider $3x^2 - 12 = 0$

$\therefore$ $x^2 - 4 = 0 \Rightarrow x^2 = 4 \Rightarrow x = -2, 2$ which are rational numbers

$\therefore$ the equation $3x^2 - 12 = 0$ has a solution in rationals is true.

**Example 1.10 :** Use quantifier to say that $\sqrt{3}$ is not a rational number.

**Sol. :** $\sim (\exists x)_Q (x^2 = 3)$.

## 1.1.9 Summary

In this lesson, we have discussed about propositions and its basic operations such as conjunction, disjunction, negation, conditional statements and bi-conditional statements. We have also defined an argument and checked whether its logically valid or invalid. An important structure i.e. quantifier and its types are also discussed in brief. We tried to elaborate the concepts with the help of suitable examples.

## 1.1.10 Key Concepts

Proposition, Logical statement, Simple statement, Compound statement, conjunction, disjunction, negation, conditional statements, bi-conditional statements, Argument, Tautology, Contradiction, Truth Table, Quantifier.

## 1.1.11 Long Questions

1. Prove that $(p \leftrightarrow q) \leftrightarrow r = p \leftrightarrow (q \leftrightarrow r)$
2. Prove that $p \rightarrow (\sim q \vee r) \equiv (p \wedge q) \rightarrow r$.
3. Test the Validity of : "If my borher stands first in the class, I will give him a watch. Either he stood first or I was out of station. I did not give my brother a watch this time. Therefore I was out of station."

## 1.1.12 Short Questions

1. Write down the truth table for

   (i) $p \wedge (q \rightarrow p)$ (ii) $p \vee (q \wedge r) \leftrightarrow (p \vee q) \wedge (p \vee r)$

   (iii) $[p \rightarrow (q \vee r)] \vee [p \leftrightarrow \sim r]$
2. If p stands for the statement, 'I like tennis', and q stands for the statement, 'I like football', then what does $\sim p \wedge \sim q$ stand for ?

## 1.1.13 Suggested Readings

1. Dr. Babu Ram, Discrete Mathematics
2. C.L. Liu, Elements of Discrete Mathematics (Second Edition), McGraw Hill, International Edition, Computer Science Series, 1986.
3. Discrete Mathematics, S. Series.
4. Kenneth H. Rosen, Discrete Mathematics and its Applications, McGraw Hill Fifth Ed. 2003.

**LESSON NO. 1.2** **Author : Dr. Chanchal**
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# SET THEORY

**Structure :**



## 1.2.1 Objectives

In this lesson, our prime objectives are:

- To study sets, operations on sets with their illustration using Venn diagram and fundamental laws of set theory
- To study about an important relation on sets called Cartesian product of sets, which is required to understand relations
- To understand how a set can be partitioned into non-overlapping subsets

## 1.2.2 Introduction to Some Basic Terms in Set Theory

Firstly, we introduce a set as :

**Def. Set :** A set is a well defined collection of distinct objects.

The word 'well defined' implies that we are given a rule with the help of which we can say whether a particular object belongs to the set or not. The word 'distinct' implies that repetition of objects is not allowed. Each object of the set is called an element of the set. Further, sets are generally denoted by capital letters A,B,C,..... while elements of the sets are denoted by small letters a,b,c, ........ .

**For Example :** (i) The set of days of a week.

(ii) The set of even integers.

A set can be represented in two ways :

(1) Tabular or Roster Method

(2) Set-builder or Rule Method

In roster form, we represent a set by listing all its elements within curly brackets { }, separated by commas while in the set-builder form, we do not list the elements but the set is represented by specifying the defining property.

| **For Example :** Set | Roster form | Set-builder form |
|---|---|---|
| (1) A set of vowels | A = {a, e, i, o, u} | A = {x : x is a vowel of english alphabet} |
| (2) A set of positive even integers upto 10 | A = {2, 4, 6, 8, 10} | A = {x : x is a positive even integer and x ≤ 10} |

There are some basic mathematical sets, such as

**N =** Set of all natural numbers

**W =** Set of all whole numbers

**I a Z =** Set of all integers

**Q =** Set of all rational numbers

**R =** Set of all real numbers

**Membership of a Set :** If an object x is a member of the set A, we write $x \in A$, which can be read as 'x belongs to A' or A contains x. Similarly, we write $x \notin A$ to show that x is not a member of the set A.

**For Example :** Let A = {1, 2, 4, 6, 7}. Here $2 \in A$ but $5 \notin A$.

**Finite Set :** A set is said to be finite if it has finite no. of elements.

**For Example :** A = {2, 4, 6, 8}

**Infinite Set :** A set is said to be infinite if it has an infinite number of elements.

**For Example :** A = {1,2, 3.......} and B = {x : x is an odd integer} are infinite sets

**Singleton Set :** A set containing only one element is called a singleton or a unit set.

**For Example :** A = {x : x is a perfect square and 30 ≤ x ≤ 40} = {6}

**Empty, Null or Void Set :** A set which contains no element, is called a null set and is denoted by $\phi$ (read as phi).

**For Example :** $A = \left\{x : x \text{ is a positive integer satisfying } x^2 = \frac{1}{4}\right\}$

**Sub-Set, Super-Set :** If every member of a set A is a member of a set B, then A is called sub-set of B and B is called super-set of A.

or if $x \in A \Rightarrow x \in B$, then A is a sub set of B and B is a super set of A and we write it $A \subset B$ which means A is contained in B or B contains A.

**Note 1.** Since every element of A belongs A

$\therefore$ $A \subset A \Rightarrow$ every set is sub set of itself.

**2.** The empty set $\phi$ is taken as a sub-set of every set.

**For Example :** Let A = {1, 2, 3, 4, 5, 6, 8, 10}, B = {2, 4, 6, 10}, = C = {1, 2, 7, 8}.

Now every element of B is an element of A, $\therefore B \subset A$

Again $7 \in C$, but $7 \notin A$

$\therefore$ $C \not\subset A$ i.e., C is not a sub-set of A.

**Equality of Sets :** Two sets A and B are said to be equal if both have the same elements. In other words, two sets A and B are equal when every element of A is an element of B and every element of B is element of A.

i.e., if $A \subset B$ and $B \subset A$, then A = B.

**For Example :** A = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10} and B = {x : x is a natural number and $1 \le x \le 10$}

Hence, A = B.

**Proper Sub-set :** A non-empty set A is said to be a proper subset of B if $A \subset B$ and $A \neq B$.

**Note : (i)** $\phi$ and A are called improper subsets of A.

**(ii)** If A has n elements, then number of subsets of A is $2^n$.

**Power set :** The power set of a finite set is the set of all sub-sets of the given set. Power set of A is denoted by P(A).

**For Example :** Take A = {1, 2, 3}

$\therefore$ P(A) = {$\phi$, {1}, {2}, {3}, {1, 2}, {1, 3}, {2, 3}, {1, 2, 3}}

**Universal Set :** If all the sets under consideration are sub-sets of a fixed set U, then U is called a universal set.

**For Example :** In Plane geometry, the universal set consists of points in a plane.

**Comparable and Non-comparable Sets :** Two sets are said to be comparable if one of the two sets is a sub-set of the other.

**For Example :** Let A = {2, 3, 5}, B = {2, 3, 5, 6}.

Here $A \subset B$, so A and B are comparable sets.

**Order of a Finite Set :** The number of different element of a finite set A is called the order of A and is denoted by O(A).

**Cardinality :** Number of different elements in a set is known as its cardinality.

**For Example :** If A = {2, 3, 6, 8}, then O(A) = 4

**Equivalent Sets :** Two finite sets A and B are said to be equivalent sets if the total number of elements in A is equal to the total number of elements in B.

**For Example :** Let A = {1, 2, 3, 4, 6}, B = {1, 2, 7, 9, 12}

∴ O(A) = 5 = O(B) ⇒ A and B are equivalent sets, denoted as A ~ B.

### 1.2.3 Operations on Sets

In order to represent various operations on sets, we use a special type of diagrams, called venn diagrams defined as :

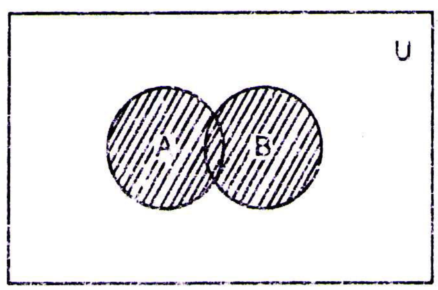
**Venn Diagrams :** The relations between sets can be illustrated by certain diagrams called **Venn diagrams.** In a Venn diagram, universal set U is represented by a rectangle and any sub-set of U is represented by a circle within a rectangle U.

Now, various operations of set theory are discussed below :

#### 1.2.3.1 Union of Two Sets

If A and B be two given sets, then their union is the set consisting of all the elements of A together with all the elements in B. We should not repeat the elements. The union of two sets A and B is written as A ∪ B.

In symbols, A ∪ B = {x : x ∈ A or x ∈ B}



A U B IS SHIELDED LIKE

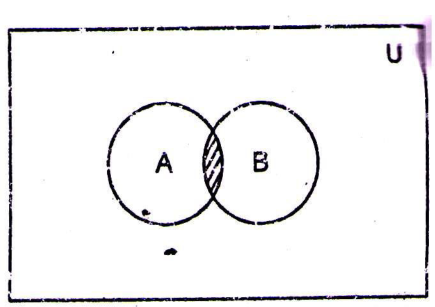
**For Example :** Let A = {1, 2, 3, 5, 8}, B = {2, 4, 6}

∴ A ∪ B = {1, 2, 3, 4, 5, 6, 8}

#### 1.2.3.2 Intersection of two sets

The intersection of two sets A and B, denoted by A ∩ B, is the set of elements common to A and B.

In symbols, A ∩ B = {x : x ∈ A and x ∈ B}



A ∩ B IS SHIELDED LIKE

**For Example :** Let A = {2, 4, 6, 8, 10, 12}, B = {2, 3, 5, 7, 11}

∴ $A \cap B = \{2\}$

**Disjoint Sets :** If A and B are two given sets such that $A \cap B = \phi$, then the sets A and B are said to be disjoint.

**For Example :** Let A = {a, b, c, d}, B = {l, m, n, p},

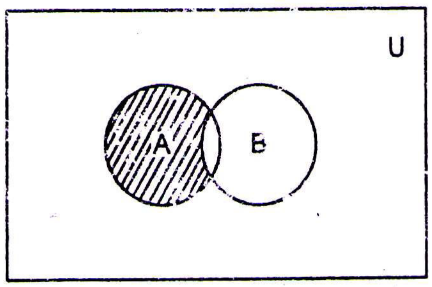
∴ $A \cap B = \phi \Rightarrow$ A and B are disjoint sets.

**1.2.3.3 Difference of two sets**

The difference of two sets A and B is the set of those elements of A which do not belong to B. We denote this by A – B.

In symbols, we write $A - B = \{x : x \in A \text{ and } x \notin B\}$

A – B is also sometimes written as A/B.




A – B IS SHIELDED LIKE ▨

**For Example :** Let A = {a, b, c, d, e}, B = {c, d, e, f, g}

Then, A – B = {a, b}

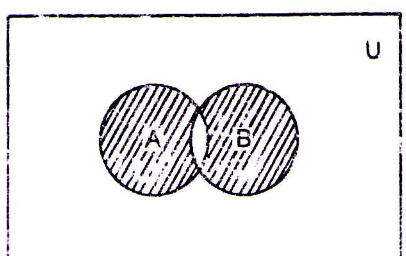
**Note.** $B - A \neq A - B$

**Symmetric Difference of Two Sets**

If A and B are any two sets, then the set $(A - B) \cup (B - A)$ is called symmetric difference of A and B and is denoted by $A \Delta B$.

In symbols, we write

$A \Delta B = \{x : (x \in A \text{ and } x \notin B) \text{ or } (x \in B \text{ and } x \notin A)\}$



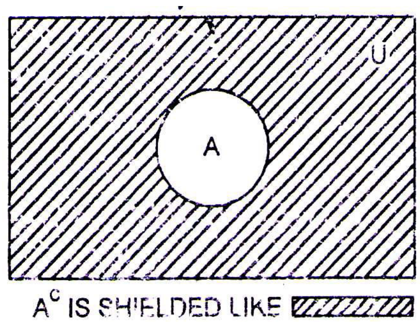

A Δ B IS SHIELDED LIKE ▨

**For Example :** Let A = {1, 2, 4}, B = {1, 2, 3, 4, 6}

$\therefore$ $A \Delta B = (A/B) \cup (B/A) = (A - B) \cup (B - A) = \{4\} \cup \{3, 5, 6\} = \{3, 4, 5, 6\}$.

**1.2.3.4 Complement of a Set**

Let A be a subset of universal set U. Then the complement of A is the set of all those elements of U which do not belong to A and we denote complement of A by $A^c$ or A'. We can write $A^c = \{x : x \in U, x \notin A\}$




**For Example :** If U = {2, 4, 6, 8,10}, A = {4, 8} then $A^c$ = {2, 6, 10}

**Note.** $U^c = \phi$ and $\phi^c = U$, $(A^c)^c = A$.

**1.2.4 Some Fundamental Laws of Algebra of Sets**

**I. Idempotent Laws**

**Statement :** If A is any set, then (i) $A \cup A = A$ (ii) $A \cap A = A$

**Proof :** (i) L.H.S. = $A \cup A$

$= \{x : x \in A \cup A\} = \{ x : x \in A \text{ or } x \in A\}$

$= \{x : x \in A\} = A$

= R.H.S.

(ii) Do Yourself.

**II. Identity Laws**

**Statement.** If A is any set, then (i) $A \cup \phi = A$ (ii) $A \cap U = A$

**Proof :** (i) L.H.S. = $A \cup \phi = \{x : x \in A \cup \phi\}$

$= \{x : x \in A \text{ or } x \in \phi\} = \{x : x \in A\}$

= A = R.H.S.

(ii) Do Yourself.

**III. Commutative Laws**

**Statement.** If A and B are any two sets, then (i) $A \cup B = B \cup A$ (ii) $A \cap B = B \cap A$

**Proof :** Do Yourself.

**IV. Associative Laws**

**Statement.** If A, B and C are any three sets, then

(i) $A \cup (B \cup C) = (A \cup B) \cup C$ (ii) $A \cap (B \cap C) = (A \cap B) \cap C$

**Proof :** Do Yourself.

**V. Distributive Laws**

**Statement.** If A, B, C are any three sets, then

(i) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

(ii) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

**Proof :** L.H.S. $= A \cup (B \cap C)$

$= \{x : x \in A \cup (B \cap C)\}$

$= \{x : x \in A \text{ or } x \in (B \cap C)\}$

$= \{x : x \in A \text{ or } (x \in B \text{ and } x \in C)\}$

$= \{x : (x \in A \text{ or } x \in B) \text{ and } (x \in A \text{ or } x \in C)\}$

$= \{x : x \in (A \cup B) \text{ and } x \in (A \cup C)\}$

$= \{x : x \in (A \cup B) \cap (A \cup C)\}$

$= \{(A \cup B) \cap (A \cup C)\}$

= R.H.S.

$\therefore$ $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

**Note.** We can also prove above result by showing that

$A \cup (B \cap C) \subset (A \cup B) \cap (A \cup C)$ and $(A \cup B) \cap (A \cup C) \subset A \cup (B \cap C)$

(ii) Do Yourself.

**VI. De Morgan's Laws**

**Statement.** If A and B are two sub-sets of U, then

(i) $(A \cup B)^c = A^C \cap B^c$ (ii) $(A \cap B)^c = A^c \cup B^c$

**Proof :** (i) L.H.S. $= (A \cup B)^c = \{x : x \in (A \cup B)^c\}$

$= \{x : x \notin (A \cup B)\}$

$= \{x : x \notin A \text{ and } x \notin B\}$

$= \{x : x \in A^c \text{ and } x \in B^c\}$

$= \{x : x \in (A^c \cap B^c)\}$

$= A^c \cap B^c =$ R.H.S.

$\therefore$ $(A \cup B)^c = A^c \cap B^c$

(ii) Do Yourself.

**1.2.5 Some Important Examples**

**Example 1 :** Let U be the set of integers and let A = {x : x is divisible by 3}, let B = {x : x is divisible by 2}. Let C = {x : x is divisible by 5} Find the elements in each of the following set :

(a) $A \cap B$ (b) $A \cup C$ (c) $A \cap (B \cup C)$ (d) $(A \cap B) \cup C$

(e) $A^c \cap B^c$ (f) $(A \cap B)^c$ (g) B/A (h) A/B (i) A/(B/C)

**Sol. :** $U = \{0, \pm 1, \pm 2, \pm 3, \pm 4, \ldots\ldots\ldots\}$

A = {x : x is divisible by 3} $= \{x : x = 3n, n \in I\} = N_3$

B = {x : x is divisible by 2} $= \{x : x = 2n, n \in I\} = N_2$

$C = \{x : x \text{ is divisible by } 5\} = \{x : x = 5n, n \in I\} = N_5$

(a) $A \cap B = N_3 \cap N_2 = N_6 \quad \because \text{l.c.m. } \{2, 3\} = 6$

$= \{\ldots\ldots\ldots\ -12, -6, 0, 6, 12, \ldots\ldots\}$

(b) $A \cup C = N_3 \cup N_5 = \{\ldots\ldots\ -9, -6, -5, -3, 0, 3, 5, 6, \ldots\ldots\}$

(c) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C) = (N_3 \cap N_2) \cup (N_3 \cap N_5) = N_6 \cup N_{15}$

$= \{\ldots\ldots -15, -12, -6, 0, 6, 12, 15 \ldots\ldots\}$

(d) $(A \cap B) \cup C = \{N_3 \cap N_2\} \cup N_5 = N_6 \cup N_5$

$= \{\ldots\ldots\ -12 - 10, 6, -5, 0, 5, 6, 10, 12, \ldots\ldots\}$

(e) $A^c \cap B^c = (A \cup B)^c = (N_3 \cup N_2)^c$

$= \{\ldots\ldots\ -11, -7, -5, -1, 1, 5, 7, 11, \ldots\ldots\}$

(f) $(A \cap B)^c = (N_3 \cap N_2)^c = (N_6)^c$

$= \{-7, -5, -3, -2, -1, 1, 2, 3, 4, 5, \ldots\ldots\}$

(g) $A/B = N_3/N_2 = N_3 - N_2$

$= \{\ldots\ldots - 10, -8, -4, -2, 2, 4, 8, 10 \ldots\ldots\}$

(h) $B/A = N_2/N_3 = N_2 - N_3$

$= \{\ldots\ldots\ -15, -9, -3, 3, 9, 15, \ldots\ldots\}$

(f) $A/(B/C) = (A/B) \cup (A \cap C)$

$= (N_3/N_2) \cup (N_3 \cap N_5) = (N_3/N_2) \cup N_{15}$

$= \{\ldots\ldots, -15, -9, -3, 3, 9, 15, \ldots\ldots\} \cup \{\ldots\ldots\ 30, 15, 0, 15, 30 \ldots\ldots\}$

$= \{\ldots\ldots, -15, -9, -3, 3, 9, 15, \ldots\ldots).$

**Example 2 :** Prove that $A \cup (B/A) = A \cup B$.

**Sol. :** L.H.S. $= A \cup (B/A) = A \cup (B - A)$

$= A \cup (B \cap A^c)$ [ $A - B = A \cap B^c$]

$= (A \cup B) \cap (A \cup A^c)$ [Distributive Law]

$= (A \cup B) \cap X$ [$A \cup A^c = X$]

$= A \cup B =$ R.H.S.

**Example 3 :** Let $A = \{1, 2, 4\}$, $B = (4, 5, 6\}$,

Find $A \cup B$, $A \cap B$ and $A - B$.

**Sol.** $A = \{1, 2, 4\}$ and $B = \{4, 5, 6\}$

(i) $A \cup B = \{1, 2, 4\} \cup \{4, 5, 6\} = \{1, 2, 4, 5, 6\}$

(ii) $A \cap B = \{1, 2, 4\} \cap \{4, 5, 6\} = \{4\}$

(iii) $A - B = \{1, 2, 4\} - (4, 5, 6\} = \{1, 2\}$.

**Example 4 :** Prove that $A \cup B = A \cap B$ iff $A = B$

**Sol. :** (i) Assume that $A \cup B = A \cap B$ ... (1)

Let x be any element of A

$\therefore$ $x \in A \Rightarrow x \in A \cup B \Rightarrow x \in A \cap B$ [$\because$ of (1)]

$\Rightarrow x \in B$

$\therefore$ $x \in A \Rightarrow x \in B$

$\therefore$ $A \subset B$

Similarly, $B \subset A$ … (2)

From (2) and (3), A = B. … (3)

$\therefore$ $A \cup B = A \cap B \Rightarrow A = B$

(ii) Assume that A = B

$\therefore$ $A \cup B = A \cup A = A$

$A \cap B = A \cap A = A$

$\therefore$ $A \cup B = A \cap B$

$\therefore$ $A = B \Rightarrow A \cup B = A \cap B$.

**Example 5 :** For any two sets A and B, prove that $A \cap B = \phi \Rightarrow A \subset B^c$.

**Sol. :** We are given that $A \cap B = \phi$ … (1)

Let x be any element of A

$\therefore$ $x \in A \Rightarrow x \notin B$ [$\because$ of (1)]

$\Rightarrow$ $x \in B^c$

$\therefore$ $x \in A \Rightarrow x \in B^c$

But x is any element of A

$\therefore$ every element of A is an element of $B^c$

$\therefore$ $A \subset B^c$.

**Example 6 :** Prove that $A^c/B^c = B/A$

**Sol. :** L.H.S. = $A^c/B^c$

$= A^c - B^c = \{x : x \in (A^c - B^c)\} = \{x : x \in A^c \text{ and } x \notin B^c\}$

$= \{x : x \notin A \text{ and } x \in B\} = \{x : x \in B \text{ and } x \notin A\}$

$= \{ x : x \in (B - A)\} = B - A = B/A$

= R.H.S.

$\therefore$ $A^c/B^c = B/A$.

**Example 7 :** Show that $A \cap (B - C) = (A \cap B) - (A \cap C)$

**Sol. :** R.H.S. $= (A \cap B) - (A \cap C)$

$= (A \cap B) \cap (A \cap C)^c = (A \cap B) \cap (A^c \cup C^c)$

$= [(A \cap B) \cap A^c] \cup [A \cap B) \cap C^c]$

$= [(A \cap A^c) \cap B] \cup [A \cap (B \cap C^c)] = (\phi \cap B) \cup [A \cap (B - C)]$

$= \phi \cup [A \cap (B - C)] = A \cap (B - C) =$ L.H.S.

### 1.2.6 Cartesian Product of Sets

**Ordered-Pair :** By an ordered pair of elements, we mean a pair (a, b) such that $a \in A$ and $b \in B$. The ordered pairs (a, b), (b, a) are different unless a = b. Also} (a, b) = (c, d) iff a = c, b = d.

**Cartesian Product of Two Sets :** The set of all ordered pairs (a, b) of element $a \in A$, $b \in B$ is called the cartesian product of the sets A and B and is denoted by $A \times B$.

In symbols, $A \times B = \{(a, b) : a \in A, b \in B\}$

**Note 1.** $A \times B$ and $B \times A$ are different sets if $A \neq B$.

**2.** $A \times B = \phi$ when one or both of A, B are empty.

**Art 1 :** Prove that

(i) $A \times (B \cup C) = (A \times B) \cup (A \times C)$

(ii) $A \times (B \cap C) = (A \times B) \cap (A \times C)$

**Proof :** (i) L.H.S. $= A \times (B \cup C)$

$= \{(x, y) : x \in A \text{ and } y \in (B \cup C)\}$

$= \{(x, y) : x \in A \ (y \in B \text{ or } y \in C)\}$

$= \{(x, y) : (x \in A \text{ and } y \in B) \text{ or } (x \in A \text{ and } y \in C)\}$

$= \{(x, y) : (x, y) \in (A \times B) \text{ or } (x, y) \in (A \times C)\}$

$= \{(x, y) : (x, y) \in (A \times B) \cup (A \times C)\}$

$= (A \times B) \cup (A \times C) =$ R.H.S.

$\therefore \quad A \times (B \cup C) = (A \times B) \cup (A \times C)$

(ii) L.H.S. $= A \times (B \cap C)$

$= \{(x, y) : x \in A \text{ and } y \in (B \cap C)\}$

$= \{(x, y) : x \in A \text{ and } (y \in B \text{ and } y \in C)\}$

$= \{(x, y) : (x \in A \text{ and } y \in B) \text{ and } (x \in A \text{ and } y \in C)\}$

$= \{(x, y) : (x, y) \in (A \times B) \text{ and } (x, y) \in (A \times C)\}$

$= \{(x, y) : (x, y) \in (A \times B) \cap (A \times C)\}$

$= (A \times B) \cap (A \times C) =$ R.H.S

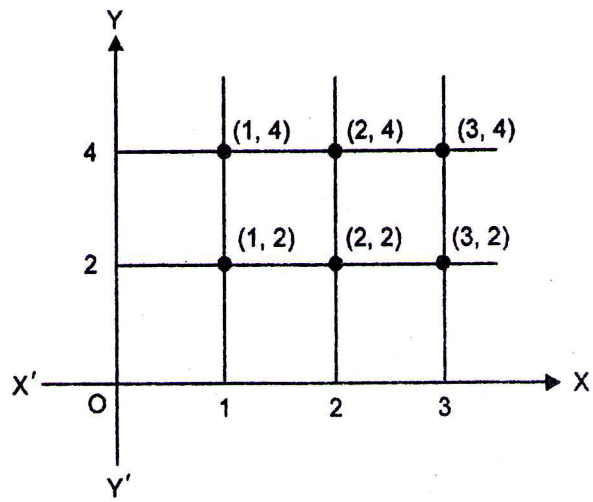
$\therefore \quad A \times (B \cap C) = (A \times B) \cap (A \times C)$.

**Example 8 :** Let A = {1, 2, 3}, B = {2, 4}. Find $A \times B$ and show it graphically.

**Sol.** Here A = {1, 2, 3}, B = {2, 4}

$A \times B = \{1, 2, 3\} \times \{2, 4\} = \{(1, 2), (1, 4), (2, 2), (2,4), (3, 2), (3, 4)\}$.

Now to represent (1, 2), we draw a vertical line through 1 and a horizontal line through 2. These two lines meet in the point which represents (1, 2). Similarly we can represent the other points in A × B and get the graphical representation of A × B.




**Example 9 :** A, B, C are any three sets, then prove that

$(A \cap B) \times C = (A \times C) \cap (B \times C)$

**Sol. :** L.H.S. $= (A \cap B) \times C$

$= \{(x, y) : x \in (A \cap B) \text{ and } y \in C\}$

$= \{(x, y) : (x \in A \text{ and } x \in B) \text{ and } y \in C\}$

$= \{(x, y) : (x \in A \text{ and } y \in C) \text{ and } (x \in B \text{ and } y \in C)\}$

$= \{(x, y) : (x, y) \in (A \times C) \text{ and } (x, y) \in (B \times C)\}$

$= \{(x, y) : (x, y) \in (A \times C) \cap (B \times C)\}$

$= (A \times C) \cap (B \times C) =$ R.H.S.

### 1.2.7 Partition of Sets

A partition of a non-empty set A is a collection $P = \{A_1, A_2, A_3, \ldots\ldots\ldots\ldots\}$ of subsets of A if and only if

(i) $A = A_1 \cup A_2 \cup A_3 \cup \ldots\ldots\ldots\ldots$

and (ii) $A_i \cap A_j = \phi$ for $i \neq j$

$A_1, A_2, A_3, \ldots\ldots$ are called cells or blocks of the partition P.

**For Example :** (i) Let A = {a, b, c} be any set.

Then, $P_1 = \{\{a\}, \{b\}, \{c\}\}$, $P_2 = \{\{a\}, \{b, c\}\}$, $P_3 = \{\{b\}, \{a, c\}\}$,

$P_4 = \{\{c\}, \{a, b\}\}$, $P_5 = \{\{a, b, c\}\}$ are partitions of the set A.

(ii) Let **Z** = set of integers. Then the collection

$P = \{\{n\} : n \in \mathbf{Z}\}$ is a partition of **Z**.

**Minimum Set or Minset or Minterm :**

Let A be any non-empty set and $B_1, B_2, \ldots\ldots\ldots, B_n$ be any subsets of A. Then the minimum set generated by the collection $\{B_1, B_2, \ldots\ldots, B_n\}$ is a set of the type $D_1 \cap D_2 \cap \ldots\ldots\ldots \cap D_n$, where each $D_1, D_2, \ldots\ldots\ldots, D_n$ is $B_i$ or $B_i$ for i = 1, 2, 3,......., n.

**For Example :** The minsets generated by two sets $B_1$ & $B_2$ are

$$A_1 = B_1 \cap B_2', A_2 = B_1 \cap B_2, A_3 = B_1^C \cap B_2, A_4 = B_1^C \cap B_2^C.$$

**Normal form (or Canonical form) :**

A set F is said to be in minset normal (or cannonical) form when it is expressed as the union of distinct non-empty minsets or it is $\phi$

i.e., either $F = \phi$ or $F = \bigcup_{\lambda \in \Delta} A_i$, where $A_\lambda$ 's are non-empty minsets.

**Principle of Duality for Sets :**

Let S be any identity in set theory involving the operation union ($\cup$), intersection ($\cap$). Then the statement S* obtained from S by changing union to intersection to union and empty set $\phi$ to universal set U is also an identity called the dual of the statement S.

**Remark :** The number of minsets generated by n sets is $2^n$.

### 1.2.8 Summary

In this lesson, we have defined about sets, subsets, equal and equivalent sets, power set, universal set, order of a set etc.Further, we have illustrated the basic operations on sets using Venn diagrams. Some fundamental laws of set theory are stated and their proofs have been discussed. Moreover, the concepts of partitioning of sets, principal of duality have been elaborated.

### 1.2.9 Key Concepts

Set, Tabular form, Set-builder form, Singleton set, Finite set, Infinite set, Void set, Subset, Superset, Power set, Universal set, Equal sets, Equivalent sets, Comparable sets, Non-comparable sets, Order, Cardinality, Union, Intersection, Complement, Difference, Symmetric difference, Venn diagram, Disjoint sets, Idempotent law, Identity law, Commutative law, Associative law, Distributive laws, De-Morgan’s laws, Cartesian product, Partitioning, Ordered pair, Minset, Normal form, Duality.

### 1.2.10 Long Questions

1. If A and B be non-empty subsets, then show that $A \times B = B \times A$ iff $A = B$.

### 1.2.11 Short Questions

1. If A, B are two sets, then show that $A \cup B = \phi \Leftrightarrow A = \phi, B = \phi$.
2. Is it true that power set of $A \cup B$ is equal to union of power sets of A and B ? Justify.
3. Prove the following :

   (i) $A \cap (A^C \cup B) = A \cap B$

   (ii) $A - (B \cap C) = (A-B) \cup (A-C)$

   (iii) $A - (B \cup C) = (A - B) \cap (A - C)$

   (iv) $A \cap (B - C) = (A \cap B) - (A \cap C)$
4. Let A = [+, –], and B = [00, 01, 10, 11]

(a) List the elements of A × B

(b) How many elements do $A^4$ and $(A \times B)^3$ have ?

### 1.2.12 Suggested Readings

1. Dr. Babu Ram, Discrete Mathematics
2. C.L. Liu, Elements of Discrete Mathematics (Second Edition), McGraw Hill, International Edition, Computer Science Series, 1986.
3. Discrete Mathematics, S. Series.
4. Kenneth H. Rosen, Discrete Mathematics and its Applications, McGraw Hill Fifth Ed. 2003.

**LESSON NO. 1.3**

**Author : Dr. Chanchal**

**LAST UPDATED ON MAY, 2023**

# FUNCTIONS

**Structure :**



## 1.3.1 Objectives

In this lesson, we are going to study :

- functions alongwith bijective function to discuss its inverse.
- composition of functions.

## 1.3.2 Introduction

Firstly, we define a function as :

**Def. Function :** Let X and Y be two non-empty sets. A subset f of X × Y is called a function from X to Y if for each $x \in X$, there exists a unique $y \in Y$ such that $(x, y) \in f$ or $f(x) = y$. It may also be defined as a rule f which associates each element of X with a unique element of Y. It is denoted by $f : X \rightarrow Y$ or $X \xrightarrow{f} Y$. Here, the set X is called the domain of f and is written as $D_f = X$. The set Y is called co-domain of f. If an element $y \in Y$ is associated with an element x of X under the rule f, then y is called the image of x under the rule f, denoted by f(x). The set consisting of images of all the elements of X under f is called Image set or Range of f and is written as $R_f$. Mathematically, $R_f = \{y : y = f(x) \text{ where } x \in X\} = f(X)$ clearly, $f(X) \subset Y$.

**Remarks :** (i) Functions are also called mappings or transformations.

(ii) To every x ∈ X, ∃ a unique y ∈ Y such that y = f (x). The unique element y ∈ Y is also called the value of f at x and is denoted by f (x).

(iii) Different elements of X may be associated with the same element of Y.

(iv) There may be elements of Y which are not associated with any element of X.

**For Example :** (i) The rule shown in the figure 3.1 is not a function as each element of X is not asociated. Here 5 ∈ X has no image in Y.

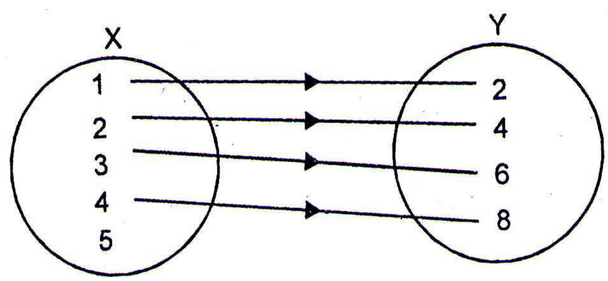



**Fig. 1.3.1**

(ii) The rule shown in the figure 3.2 is not a function as I ∈ X is associated with more than one element namely a and b of Y.

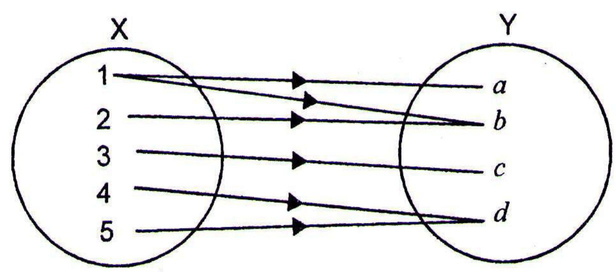



**Fig. 1.3.2**

(iii) The rule shown in the figure 3.3 is a function as each element of X is associated with a unique element of Y.

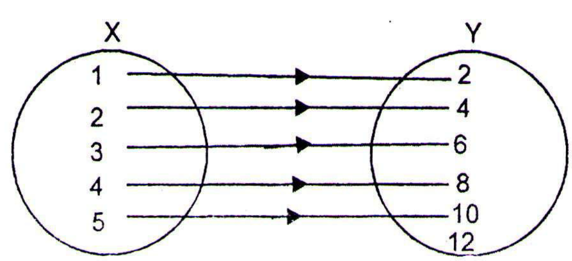



**Fig. 1.3.3**

### 1.3.3 One-One and Onto Functions

**Def. One-One function or Injective function :** A function f from X to Y is said to be one-one (abbreviated 1-1) iff

$x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2) \ \forall \ x_1, x_2 \in X$, or equivalently

$f(x_1) = f(x_2) \Rightarrow x_1 = x_2 \ \forall \ x_1, x_2 \in X$.

In other words, if different elements of X under the rule f have different images in Y, then f is called one-one function. A function which is not 1-1 is called many-one function.

**For Example :**

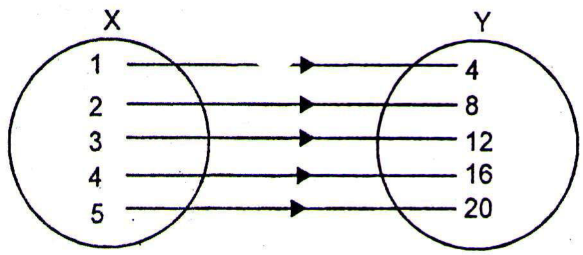



**Fig. 1.3.4**

**Def. Onto function a Surjective Function :**

**Def.** A function from f from X to Y is called onto iff every element of Y is an image of at least one element of X. Otherwise f is called an into mapping.

**Note.** In the case of onto function, $R_f = Y$, while in the case of into function $R_f$ is a proper subset of Y.

**For Example :** (i) The function f depicted in the diagram 3.5 is one-one onto.

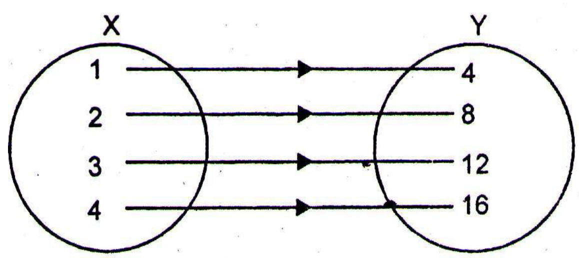



**Fig. 1.3.5**

(ii) Let X = {1, 2, 3, 4, 5, 6}, Y = {2}

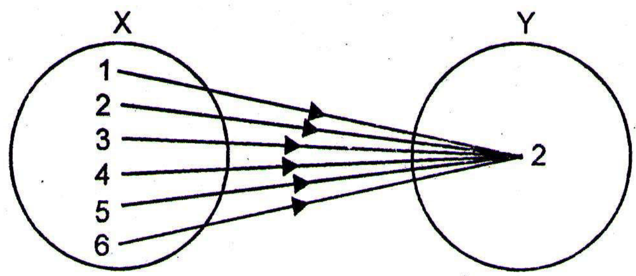



**Fig. 3.6**

**Def. Bijective function :** A function which is one-one and onto is called bijective function. It is also called **one-one correspondence.**

**For Example :** The function shown in fig. 3.5 is a bijective function.

### 1.3.4 Types of Functions

There are various types of functions as disussed below :

**I. Real valued function on real variables**

Let X, Y be two non-empty subsets of real numbers. Then, every function f from X to Y is called a real valued function on real variables.

**II. Equal functions**

Two real valued functions f and g are said to be equal iff $D_f = D_g$ and

$f(x) = g(x)\ \forall\ x \in D_f$. We write it as f = g.

**III. Constant Function**

A function $f : X \rightarrow Y$ is called a constant function if $f(x) = k$ for every $x \in X$ and Here $k \in Y$ is fixed.

Function shown in figure 3.6 is a constant function.

**IV. Identity Mapping**

Let $I_x : X \rightarrow X$ be defined by, $I_x(x) = x\ \forall\ x \in X$.

Then $I_x$ is called the identity mapping on X.

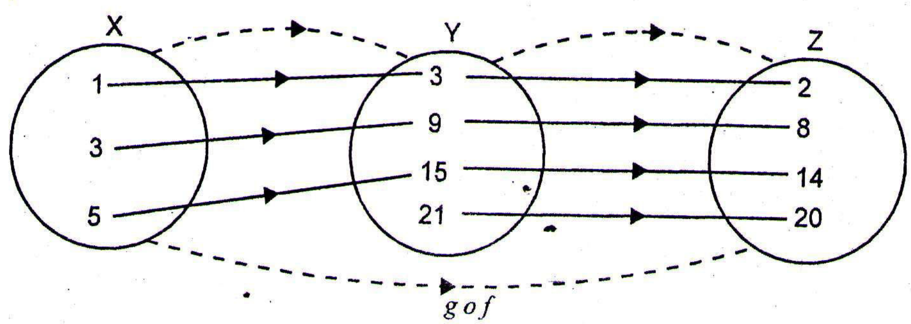
**V. Inverse Mapping**

Let $f : X \rightarrow Y$ be a one-one onto mapping. Then the mapping $f^{-1} : Y \rightarrow X$ which associates to each element $y \in Y$, the unique element $x \in X$ such that $f(x) = y$ is called the inverse map of f.

### 1.3.5 Composition of Functions

**Def. :** Let f be a function with domain X and range in Y and let g be a function with domain Y and range in Z. The function with domain X and range in Z which maps an element $x \in X$, into $g(f(x))$, is called the composite of the functions f and g and is written as g of.

**For Example :** Let X = {1, 3, 5}, Y = {3, 9, 15, 21}, Z = {2, 8, 14, 20}




Let f be a function from X to Y and g be a function from Y to Z such that

f = {(1, 3), (3, 9), (5, 15)}, g = {(3, 2), (9, 8), (15, 14), (21, 20)}

then, gof = {(1, 2), (3, 8), (5, 14)}.

**Note :** (i) gof is defined only when $R_f \subset D_g$.

(ii) It is possible that one of fog and gof may be defined while the other may not be defined.

(iii) gof and fog both may be defined but may not be equal.

(iv) Let f, g, h be three functions and $\alpha$ be a real number, then

(a) (fog) oh = fo (goh) (Associative Law)

(b) fo (g+h) = fog + foh (Distributive Law)

(c) ($\alpha$ f) og = $\alpha$. (fog) (Scalar multiplication)

**Art 1 :** If f : A $\rightarrow$ B and g : B $\rightarrow$ C are both one-one and onto maps i.e. bijective maps, then gof is also bijective map.

**Proof :** Since f : A $\rightarrow$ B and g : B $\rightarrow$ C are maps, therefore gof is also a map from A to C. Let $x_1, x_2 \in A$ such that

**One–One :** (gof) $(x)_1$ = (gof) $x_2$

$\Rightarrow$ $g(f(x_1)) = g(f(x_2))$

$\Rightarrow$ $f(x_1) = f(x_2)$, since g is one-one

$\Rightarrow$ $x_1 = x_2$ since f is one-one

$\therefore$ gof is a one-one map

**Onto :** Since f, g are onto

Let c $\in$ C be any element, then $\exists$ b $\in$ B such that

g (b) = c ($\because$ g is onto)

Again for this b $\in$ B, $\exists$ some a $\in$ A such that

f (a) = b ($\therefore$ f is onto)

$\therefore$ gof (a) = g (f(a)) = g(b) = c

Thus for $c \in C$, $\exists$ $a \in A$ such that gof (a) = c

Hence gof : $A \rightarrow C$ is onto.

**Art 2 :** If f : $A \rightarrow B$ and g : $B \rightarrow C$ are two maps such that gof : $A \rightarrow C$ is both one-one and onto map, then f is one-one and g is onto.

**Proof :** Since f : $A \rightarrow B$, g : $B \rightarrow C$ are maps

$\therefore$ gof : $A \rightarrow C$ is a map. Also gof is given to be one-one map.

**f is one-one :** If possible, suppose that f is not one-one and g is one-one.

$\therefore$ $\exists$ $x_1, x_2 \in A$ such that $x_1 \neq x_2$ but $f(x_1) = f(x_2)$.

But $f(x_2) = f(x_2) \Rightarrow g(f(x_1)) = g(f(x_2))$ [$\because$ g is supposed to be one-one]

$\Rightarrow$ gof $(x_1)$ = gof $(x_2)$

$\therefore$ $x_1, x_2 \in A$ such that $x_1 \neq x_2$, but (gof) $(x_1)$ = (gof) $(x_2)$

$\therefore$ gof is not one-one, which is against the given hypothesis that gof is one-one.

Thus our supposition is wrong and f is one-one but g is not one-one.

We, now give an example to illustrate that if gof is one-one, then g may not be one-one.

Let A = {1, 2}, B = {4, 5, 6}, C = {7, 8, 9, 10}

Let f = {(1, 4), (2, 6)} and g = {(4, 7), (5, 8), (6, 8)}

then f and g are functions from A to B and from B to C respectively.

Have $R_f = \{5, 6\} \subseteq D_g = \{4, 5, 6\}$

$\therefore$ $R_f \subseteq D_g$

$\Rightarrow$ gof is defined and $D_{gof} = D_f = A = \{1, 2\}$

gof (1) = g (f(1)) = g(4) = 7 and gof (2) = g (f(2)) = g (6) = 8

$\therefore$ gof = {(1, 7), (1, 8)}

Here, gof is one-one map since different elements of A have different image

But g is not one-one since g (5) = g (6) = 8, but $5 \neq 6$.

Since f : $A \rightarrow B$ and g : $B \rightarrow C$ are maps, so gof is a map from A to C. We are given that gof : $A \rightarrow C$ is onto. We now prove that g is onto

Let $z \in C$.

**g is onto :** Since gof : $A \rightarrow C$ is onto, so $\exists$ $x \in A$ such that gof (x) = z

$\Rightarrow$ g (f (x)) = z $\Rightarrow$ g (y) = z where y = f (x)

Since $x \in A$ and f is map from A to B

Therefore $f(x) \in B \Rightarrow y \in B$

for given $z \in C$, we have determined $y \in B$ such that g (y) = z

$\therefore$ g : $B \rightarrow C$ is onto

Now, we show by an example that if gof is onto, then f may not be onto

Let A = {1, 2}, B = {4, 5, 6}, C = {7}

Let f = {(1, 4), (2, 6)} and g = {(4, 7), (5, 7), (6, 7)}

Then f is a function from A to B and g is a function from B to C

$\therefore$ gof is a function from A to C such that gof = {(1, 7), (2, 7)}

Here, g is onto. But f is not onto since 5 belonging to B has no pre-image in A under the map f but $g : B \to C$ is onto.

### 1.3.6 Invertible Function

**Def. :** A function f defined from X to Y is said to be invertible if three exists a function g from Y to X such that $gof = I_X$ and $fog = I_Y$, where $I_X$ is an identity mapping on X and $I_Y$ is an identity mapping on Y.

**Note :** f and g are called inverse of each other.

**Art 3 :** Let $f : X \to Y$. Then $fo\, I_x = f = I_Y of$.

**Proof :** Let x be any element of X and let f (x) = y, $y \in Y$

Since $f : X \to Y$ and $I_Y : Y \to Y$

$\therefore$ $I_X of : X \to Y$

Now $(I_Y of)(x) = I_Y(f(x)) = I_Y(y) = y = f(x)\ \forall\ x \in X$

$\therefore$ $I_Y of = f$

Again $I_X : X \to X$ and $f : X \to Y$

$\therefore$ $fo\, I_X : X \to Y$

Now $(fo\, I_X)(x) = f(I_X(X)) = f(x)\ \forall\ x \in X$

$\therefore$ $fo\, I_X = f$.

**Art 4 :** If $f : X \to Y$ is invertible, then its inverse is unique.

**Proof :** Let $g : Y \to X$ and $h : Y \to X$ be two inverse functions of $f : X \to Y$

$\therefore$ $fog = I_Y$, $gof = I_X$ and $foh = I_Y$, $hof = I_X$

Now $g(y) = g(I_Y(y)) = g\{(foh)(y)\} = g\{f(h(y))\}$

$= (gof)(h(y)) = I_X(h(y))$

$\therefore$ $g(y) = h(y)\ \forall\ h(y) \in X \Rightarrow g(y) = h(y)\ \forall\ y \in Y$

$\therefore$ g = h

$\therefore$ inverse of function f is unique.

**Note.** (1) Inverse of f, if it exists is denoted by $f^{-1}$.

(2) $f^{-1} of = I_X$ and $f of^{-1} = I_Y$ where $f : X \to Y$ is an invertible function.

**Art 5 :** A function $f : X \to Y$ is invertible iff f is one-one and onto.

**Proof.** (i) Assume that $f : X \to Y$ is invertible

$\therefore$ $\exists$ a function $g : Y \to X$ such that $fog = I_Y$ and $gof = I_X$

We will prove that f is one-one and onto.

**To prove that f is one-one**

Let $x_1, x_2 \in X$ and $f(x_1) = f(x_2)$

$\Rightarrow$ $g(f(x_1)) = g(f(x_2))$ $\Rightarrow$ $(gof)(x_1) = (gof)(x_2)$

$\Rightarrow$ $I_X(x_1) = I_X(x_2)$ $\Rightarrow$ $x_1 = x_2$

$\therefore$ $f(x_1) = f(x_2)$ $\Rightarrow$ $x_1 = x_2\ \forall\ x_1, x_2 \in X$

$\therefore$ f is one-one.

**To prove f is onto**

To each $y \in Y$, there exists $x \in X$ such that $g(y) = x$.

$\Rightarrow \quad f(g(y)) = f(x) \quad \Rightarrow \quad (fog)(y) = f(x)$

$\Rightarrow \quad I_Y(y) = f(x) \quad \Rightarrow \quad y = f(x)$

$\therefore$ f is onto.

(ii) Assume that $f : X \rightarrow Y$ is one-one and onto. We have to show that f is invertible.

Since f is one-one and onto

$\therefore$ to each $y \in Y$, there exists one and only one $x \in X$ such that $f(x) = y$.

$\therefore$ we can define a function $g : Y \rightarrow X$ such that $g(y) = x$ iff $f(x) = y$

Now $(gof)(x) = g(f(x)) = g(y) = x, \forall x \in X$

$\therefore \quad gof = I_x$

Again $(fog)(y) = f(g(y)) = f(x) = y, \forall y \in Y$

$\therefore \quad fog = I_y$

$\therefore$ f is invertible.

**Art 6 :** If a function $f : X \rightarrow Y$ be one-one and onto then, $f^{-1}$ is also one-one and onto.

**Proof :** $\because f : X \rightarrow Y$ is one-one and onto

$\therefore \quad f^{-1} : Y \rightarrow X$ exists and $f^{-1} of = I_X$, $fof^{-1} = I_Y$

**To prove $f^{-1}$ is one-one**

Let $y_1 \in Y, y_2 \in Y$.

Now $f^{-1}(y_1) = f^{-1}(y_2)$

$\Rightarrow \quad f(f^{-1}(y_1)) = f(f^{-1}(y_2))$

$\Rightarrow \quad (fof^{-1})(y_1) = (fof^{-1})(y_2)$

$\Rightarrow \quad I_Y(y_1) = I_Y(y_2) \quad \Rightarrow \quad y_1 = y_2$

$\therefore \quad f^{-1}(y_1) = f^{-1}(y_2) \quad \Rightarrow \quad y_1 = y_2 \ \forall y_1, y_2 \in Y$

$\therefore \quad f^{-1}$ is one-one.

**To prove $f^{-1}$ is onto**

To each $x \in X$, there exists $y \in Y$ such that $y = f(x)$

$\Rightarrow \quad f^{-1}(y) = f^{-1}(f(x)) \quad \Rightarrow \quad f^{-1}(y) = (f^{-1} of)(x)$

$\Rightarrow \quad f^{-1}(y) = I_X(x) \quad \Rightarrow \quad f^{-1}(y) = x$

$\therefore \quad f^{-1}$ is onto.

**Cor.** Since, $f^{-1}$ is invertible and its inverse is f.

$\therefore \quad (f^{-1})^{-1} = f.$

**Art 7 :** Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ and let f, g be one-one onto. Then $gof : X \rightarrow Z$ is also one-one onto and $(gof)^{-1} = f^{-1} og^{-1}$.

**Proof :** Do Yourself.

### 1.3.7 Floor and Ceiling Functions

For any real number x, the floor function of x means the greatest integer which is less than or equal to x.

It is denoted by x.

**For Example :** 2.58 = 2, –4.4 = 5, 2 = 2

For any real number x, the ceiling function of x means the least integer which is greater than or equal to x. It is denoted by x.

**For Example :** 2.58 = 3, –4.4 = –4, 2 = 2.

For any real number x, the integer function of x converts x into an integer by deleting the fractional part of x. It is denoted by INT (x).

**For Example :** INT (2.44) = 2, INT (–4.44) = –4.

**Note :** (i) If x is on integer, then x = x . Otherwise x + 1 = x

(ii) x =n $\Rightarrow$ n $\le$ x < n + 1 and [x] = n $\Rightarrow$ n–1 < x $\le$ n

(iii) INT (x) = x if x is positive and INT (x) = x if x is negative.

**1.3.8 Some Important Examples**

**Example 1 :** Let a function f : **R** $\rightarrow$ **R** be defined by f (x) = 2x + 3 $\forall$ x $\in$ R. Prove that f is one-one and onto.

**Sol.** Let $x_1, x_2 \in R$ such that $f(x_1) = f(x_2)$

$\therefore$ $2x_1 + 3 = 2x_2 + 3 \Rightarrow x_1 = x_2$

$\therefore$ f is one-one map

Let y $\in$ R. Let $y = f(x_0)$

Then $2x_0 + 3 = y \Rightarrow x_0 = \frac{y-3}{2}$

Since $y \in R$, so $\frac{y-3}{2} \in R$ i.e., $x_0 \in R$

$$f(x_0) = 2x_0 + 2 = 2\frac{y-3}{2} + 3 = y$$

Therefore for each y $\in$ R, there exists $x_0 \in R$ such that $f(x_0) = y$.

$\therefore$ f is onto.

**Example 2 :** If $f(x) = \frac{1}{1-x}$, then what is f [f {f(x)}] ?

**Sol.** Here $f(x) = \frac{1}{1-x}$

$$\therefore \quad f\{f(x)\} = \frac{1}{1-f(x)} = \frac{1}{1-\frac{1}{1-x}} = \frac{1-x}{1-x-1} = \frac{1-x}{-x}$$

$$\therefore \qquad f[f\{f(x)\}] = \frac{1-f(x)}{-f(x)} = \frac{1-\frac{1}{1-x}}{\frac{-1}{1-x}} = \frac{1-x-1}{-1} = \frac{-x}{-1}$$

$\Rightarrow \qquad f[f\{f(x)\}] = x.$

**Example 3 :** Let f and g be two functions from **IR** $\to$ **R** defined by f (x) = $x^2 + 3x + 2$ and g (x) = 4x – 1. Find fog and gof. Also calculate (gof) (–1) and fog (–1). Is composition commutative or not ?

**Sol.** gof is a map from A to C defined by gof (x) = g (f (x)) $\forall$ x $\in$ A.

Here, f : **R** $\to$ **R** is defined by f (x) = $x^2 + 3x + 2$, $\forall$ x $\in$ R,

and g : **R** $\to$ **R** is defined by g (x) = 4x – 1, $\forall$ x $\in$ R

$\therefore$ fog : **R** $\to$ **R** is defined by

fog (x) = f (g (x)) = f (4x + 1) = $(4x - 1)^2 + 3(4x - 1) + 2$

$= 16x^2 - 8x + 1 + 12x - 3 + 2$

i.e., (fog) x = $16x^2 + 4x$. ... (1)

gof : **R** $\to$ **R** is defined by

gof = g (f(x)) = g ($x^2 + 3x + 2$) = 4 ($x^2 + 3x + 2$) – 1

$= 4x^2 + 12x + 8 - 1$

gof (x) = $4x^2 + 12x + 7$ ... (2)

By (1), (fog) (–1) = $16(-1)^2 + 4(-1) = 16 - 4 = 12$

By (2), (gof) (–1) = $4(-1)^2 +| \; 12(-1) + 7 = -1$

$\therefore$ fog (–1) $\neq$ gof (–1).

Hence composition of maps is not commutative.

**Example 4 :** Is $f(x) = \frac{x-1}{x+1}$ invertible in its domain ? If so, find $f^{-1}$. Further verify that $(fof^{-1})(x) = x$.

**Sol.** Here $f(x) = \frac{x-1}{x+1}$

$D_f$ = set of all reals except – 1

$R_f$ = set of all reals except 1

Let $x_1, x_2 \in D_f$ and $f(x_1) = f(x_2)$

$\Rightarrow \quad \dfrac{x_1 - 1}{x_1 + 1} = \dfrac{x_2 - 1}{x_2 + 1} \Rightarrow x_1x_2 - x_2 + x_1 - 1 = x_1x_2 + x_2 - x_1 - 1$

$\Rightarrow \quad 2x_1 = 2x_2 \quad \Rightarrow \quad x_1 = x_2$

$\therefore \quad f(x_1) = f(x_2) \quad \Rightarrow \quad x_1 = x_2$

$\therefore$ f (x) is 1 – 1 in $D_f$.

$\forall\, y \in R_f\ \exists\, x = \dfrac{1+y}{1-y} \in D_f\ (\text{where } y \neq 1)$

$$\text{s.t. } f(x) = f\left(\frac{1+y}{1-y}\right) = \frac{\frac{1+y}{1-y} - 1}{\frac{1+y}{1-y} + 1} = \frac{\frac{1+y-1+y}{1-y}}{\frac{1+y-1-y}{1-y}} = \frac{2y}{2} = y$$

$\therefore$ the mapping fis onto

$\therefore$ f is both 1–1 and onto $\Rightarrow f^{-1}$ exists

Now to find $f^{-1}$, Let $y = f(x) = \dfrac{x-1}{x+1}$

$\therefore \quad xy + y = x - 1 \quad \Rightarrow \quad x - xy = y + 1$

$\Rightarrow \quad x(1-y) = 1 + y \Rightarrow x = \dfrac{1+y}{1-y}.$

$\therefore \quad f^{-1}(y) = \dfrac{1+y}{1-y} \Rightarrow f^{-1}(x) = \dfrac{1+x}{1-x}$

and $D_{f^{-1}} =$ Set of all reals except 1

$$\text{Verification : } (fof^{-1})(x) = f(f^{-1}(x)) = \frac{f^{-1}(x) - 1}{f^{-1}(x) + 1} = \frac{\frac{1+x}{1-x} - 1}{\frac{1+x}{1-x} + 1} = \frac{\frac{1+x-1+x}{1-x}}{\frac{1+x-1-x}{1-x}} = \frac{2x}{2} = x$$

$(fof^{-1})(x) = x$.

**Example 5 :** Prove that function f: C $\to$ R, defined by f (z) = $|z|$ is neither one-one nor onto.

**Sol.** Here, $f : C \to R$ defined by $f(z) = |z|$

Let $z_1 = 2 + 2i$, $z_2 = 2 - 3i$ $\Rightarrow$ $z_1 \neq z_2$

$$f(z_1) = |z_1| = \sqrt{4+9} = \sqrt{13} \qquad [z = x + iy, |z| = \sqrt{x^2 + y^2}]$$

$$f(z_2) = |z_2| = \sqrt{4+9} = \sqrt{13}$$

Here $f(z_1) = f(z_2)$. But $z_1 \neq z_2$.

so, f is not one-one.

**Onto :** again let $-3 \in R$.

But there does not exist any complex number such that $f(z) = -3$.

So, f is not onto.

**Example 6 :** Let X = Y = Z = R and let $f : X \to Y$ and $g : Y \to Z$ are such that $f(x) = 2x + 1$ and $g(y) = y/3$. Verify that $(gof)^{-1} = f^{-1}og^{-1}$.

**Sol.** Here $f : X \to Y$ is defined by $f(x) = 2x + 1$ and $g : Y \to Z$ is defined by $g(y) = \frac{y}{3}$

L.H.S. : $gof(x) = g(f(x)) = g(2x+1) = \frac{2x+1}{3}$

Now we find $(gof)^{-1}$, Let $gof(x) = y$

then, $y = \frac{2x+1}{3} \Rightarrow x = \frac{3y-1}{2}$

$$\Rightarrow \quad (gof)^{-1}y = \frac{3y-1}{2} \text{ or } (gof)^{-1}x = \frac{3x-1}{2}$$

R.H.S. : $f(x) = 2x + 1$

$$\Rightarrow \quad y = 2x + 1 \Rightarrow x = \frac{y-1}{2}$$

$$\Rightarrow \quad f^{-1}(y) = \frac{y-1}{2} \text{ or } f^{-1}(x) = \frac{x-1}{2}$$

Again, $g(y) = \frac{y}{3} \Rightarrow x = \frac{y}{3} \Rightarrow y = 3x$

$$\Rightarrow \quad g^{-1}(x) = 3x \text{ or } g^{-1}(y) = 3y$$

Now $f^{-1}og^{-1}(x) = f^{-1}(g^{-1}(x)) = f^{-1}(3x) = \frac{3x-1}{2}$

so $(gof)^{-1} = f^{-1} og^{-1}$.

**Example 7:** Let f and g be functions from **R** to **R** defined by f (x) = [x] and g(x) = |x|. Determine whether fog = gof.

**Sol.** Given f (x) = [x] and g(x) = |x|

fog (x) = f [g (x)] = f (|x|) = [|x|]

gof (x) = g [f(x)] = g ([x]) = |[x]|

Now fog $\neq$ gof.

As fog (–3.2) = f[g (–3.2)] = f (3.2) = 3

gof (–3.2) = g [f (–3.2)] = g (–4) = 4.

**1.3.9 Summary**

In this lesson, we have defined functions alongwith bijective function, invertible function. Further, the concept of composition of functions has been elaborated.

**1.3.10 Key Concepts**

Function, One-one function, Onto function, Bijective function, Invertible function, Identity mapping, Composition of functions, Floor function, Ceiling function.

**1.3.11 Long Questions**

1. Is function f : **IR** → **R** defined by $f(x) = \frac{1}{x}$ is bijective in its domain.

2. Let f : **IR** → **R** and g : **IR** → **IR** be real valued functions defined by f (x) = $2x^3 - 1$, x ∈ R and $g(x) = \left[\frac{1}{2}(x+1)\right]^{\frac{1}{3}}$, x ∈ **IR**. Show that f and g are bijective and each is inverse of other.

3. If f(x) = $x^2 - 1$, g (x) = 3x + 1, then describe the following functions :

   (i) gof (ii) fog (iii) gog (iv) fof

**1.3.12 Short Questions**

1. Prove that a function f : **IR** → **R**, defined by f (x) = $x^3$ is one-one onto.

2. If y = f(x) = $\frac{x+2}{x-1}$, then show that x = f(y).

3. Give an example of a map which is

   (i) one-one but not onto (ii) onto but not one-one.

### 1.3.13 Suggested Readings


1. Dr. Babu Ram, Discrete Mathematics
2. C.L. Liu, Elements of Discrete Mathematics (Second Edition), McGraw Hill, International Edition, Computer Science Series, 1986.
3. Discrete Mathematics, S. Series.
4. Kenneth H. Rosen, Discrete Mathematics and its Applications, McGraw Hill Fifth Ed. 2003.

**LESSON NO. 1.4** **Author : Dr. Chanchal**
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# STUDY OF ALGORITHMS

**Structure:**



## 1.4.1 Objectives

In this lesson, we are going to study about Algorithms and their complexity. Further growth rate functions will be introduced.

## 1.4.2 Introduction

The word algorithm comes from the name of Persian author, Abu Jafar, who wrote a book on mathematics. It has several applications and the work regarding algorithm has gained significant importance. In computer science, the **analysis of algorithms** is the determination of the amount of resources (such as time and storage) necessary to execute them or we can say that algorithms are used to design a method that can be used by the computer to find out the solution of a particular problem. Most algorithms are designed to work with inputs of arbitrary length. Usually, the efficiency or running time of an algorithm is stated as a function relating the input length to the number of steps (time complexity) or storage locations (space complexity).

Algorithm analysis is an important part of a broader computational complexity theory, which provides theoretical estimates for the resources needed by any algorithm which solves a given computational problem. These estimates provide an insight into reasonable directions of search for efficient algorithms. So, an algorithm may be defined as follows :

- An algorithm is a set of rules for carrying out calculation either by hand or on a machine.
- It is a finite step-by-step list of well-defined instructions for solving a particular problem.
- An algorithm is a sequence of computational steps that transform the input into the output.

**For example :** The algorithm described below is designed to find out the minimum of three numbers $a,b$ and $c$.

1. min= $a$
2. If $b$ < min, then min= $b$ .
3. If $c$ < min, then min= $c$ .

### 1.4.3 Various Characteristics of Algorithms

1. **Input :** The algorithm starts with an input or we can say that the algorithm receives input. The input involves the supply of one or more quantities.
2. **Output :** The algorithm ends with an output or we can say that the algorithm produces output. The result we obtain at the end is called output and at least one quantity is produced.
3. **Precision :** The steps involved in the algorithm are precisely stated. Each instruction mentioned in the algorithm should be clear and unambiguous.
4. **Determinism :** The intermediate results of each step of execution are unique and determined only by the inputs and the results of the preceding steps.
5. **Finiteness :** The number of steps in an algorithm should be finite. It means that if we trace out the instruction of an algorithm, then for all the cases, the algorithm must terminate after a finite number of steps.
6. **Correctness :** The output produced by an algorithm must be correct.
7. **Generality :** The algorithm must apply to a set of inputs.
8. **Effectiveness :** Every instruction stated in an algorithm should be very basic and clear so that it can be carried out very effectively.

### 1.4.4 Study of Algorithms

The study of algorithm involves several important and active areas of research out of which the most essential are discussed below:

1. **Creating an Algorithm :** The art of creating an algorithm can never be fully automated. By mastering these design strategies, it will become very easy to design new and useful algorithms.
2. **Algorithm Validation :** After the process of design of an algorithm, it is necessary to check that it computes the correct answer for all the possible legal inputs. This process is known as algorithm validation.
3. **Analysis of Algorithm or Performance Analysis :** As an algorithm is executed, it uses the computer's central processing unit (CPU) to perform operations and its memory to hold the program and data. Analysis of algorithms refers to the task of determining the computing time and storage that an algorithm requires and it should be done with great mathematical skills.
4. **Testing a Program :** It consists of two phases: debugging and profiling (or performance measurement). Debugging is the process of executing programs on sample data set to investigate whether faulty errors occur and, if so, correct them. Profiling is the process of executing a correct program on data sets and measuring the time and space it takes to compute the results.

### 1.4.4.1 Aspects of Algorithm Efficiency

The two important aspects of algorithm efficiency are:

I. The amount of time required to execute an algorithm and
II. The amount of memory space needed to run a program.

A computer requires a certain amount of time to carry out arithmetic operations. Moreover, different algorithms need different amount of space to hold numbers in memory for later use. An analysis of the time required to execute an algorithm of a particular size is referred to as the time complexity of the algorithm while an analysis of the computer memory required involves the space complexity of the algorithm.

Let $M$ be an algorithm and $n$ be the size of the input data. The time and space used by the algorithm are the two main features for the efficiency of $M$. The time is measured by counting the number of key operations. **For example :** In sorting and searching, one counts the number of comparisons but in arithmetic, one counts multiplications and neglects the additions.

These key operations are so defined that the time for the other operations is more than or at most proportional to the time for the key operations. The space is measured by counting the maximum of memory needed by an algorithm.

### 1.4.4.2 Some Important Functions

Functions play an important role in the study of algorithms and their analysis. Some of the important mathematical functions which are used very often in algorithms, are discussed below:

1. **Absolute Value Function :** Let $x$ be any real number. Then, the absolute value of $x$, denoted by $|x|$ may be defined as

$$|x| = \begin{cases} x, x \geq 0 \\ -x, x < 0 \end{cases}$$

For example : $|-2| = 2, |9| = 9.$

2. **Characteristic Function :** Let $A$ be any set and $S$ be any subset of $A$. Then, characteristic function denoted by $c_S = \begin{cases} 1, x \in S \\ 0, x \notin S \end{cases}$

For example : If $A = \{a, b, c\}$ and $S = \{a, b\}$, then $c_S(a) = 1, c_S(b) = 1, c_s(c) = 0$ because $a, b \in S$ and $c \notin S$. So, we can write

$c_S = \{(a,1), (b,1), (c,0)\}$.

3. **Floor Function :** For any real number $x$, the floor function of $x$ means the greatest integer which is less than or equal to $x$. It is denoted by $\lfloor x \rfloor$.

For example : $\lfloor 2.58 \rfloor = 2, \lfloor -4.4 \rfloor = -5, \lfloor 2 \rfloor = 2$.

4. **Ceiling Function :** For any real number $x$, the ceiling function of $x$ means the least integer which is greater than or equal to $x$. It is denoted by $\lceil x \rceil$.

For example : $\lceil 2.58 \rceil = 3, \lceil -4.4 \rceil = -4, \lceil 2 \rceil = 2$.

5. **Integer Function :** For any real number $x$, the integer function of $x$ converts $x$ into an integer by deleting the fractional part of $x$. It is denoted by INT($x$).
For example : INT(2.44)=2, INT(-4.44)=-4.

**Note :** (i) If $x$ is an integer, then $\lfloor x \rfloor = \lceil x \rceil$. Otherwise $\lfloor x \rfloor + 1 = \lceil x \rceil$.

(ii) $\lfloor x \rfloor = n \Rightarrow n \leq x < n+1$ and $\lceil x \rceil = n \Rightarrow n-1 < x \leq n$.

(iii) INT($x$)=$\lfloor x \rfloor$ if $x$ is positive and INT($x$)=$\lceil x \rceil$ if $x$ is negative.

6. **Remainder Function :** Let $M$ be a positive integer and $k$ be any integer. Then, $k$ (mod $M$) is called the remainder function and it denotes the integer remainder when $k$ is divided by $M$. Also, $k$ (mod $M$) is a unique integer such that $k = Mq + r$ where $0 \leq r < M$.

**Note :** (i) For positive numbers, we simply divide $k$ by $M$ to obtain remainder $r$ but for negative numbers, we divibe $|k|$ by $M$ to get remainder $r'$ and $k$ (mod $M$)=$M$ - $r'$ if $r' \neq 0$.
For example : 26(mod4)=2 and -35(mod9)=9-8=1.

7. **Logarithm and Exponent Functions :** Let $b$ be any positive integer. The logarithm of any positive number $x$ to base $b$ is written as $\log_b x$ and it represent exponent to which $b$ must be raised to obtain $x$. Mathematically, we can write $y = \log_b x$ iff $b^y = x$.

   For example : $\log_3 216 = 6$

   **Note :** (i) For any base $b$, $\log_b 1 = 0$ and $\log_b b = 1$ because $b^0 = 1$ and $b^1 = b$.

   (ii) Logarithm of a negative number and logarithm of zero is not defined.

### 1.4.5 Recursive Algorithm

A recursive algorithm is an algorithm which is used with smaller or simpler input values and which obtains the result for the current input by applying simple operations to the returned value for the smaller or simpler input. In other words, if a problem can be solved utilizing solutions to smaller versions of the same problem, and the smaller versions reduce to easily solvable cases, then one can use a recursive algorithm to solve that problem. For example, the elements of a recursively defined ser or a recursively defined function can be obtained by a recursive algorithm.

If a set or a function is defined recursively, then a recursive algorithm to compute its members or values describes the definition. Initial steps of the recursive algorithm correspond to the basis clause of the recursive definition and they identify the basis elements. It is then followed by the steps corresponding to the inductive clause, which reduce the computation for an element of one generation to that of elements of the immediately preceding generation.

### 1.4.6 Complexity of Algorithms

For an algorithm $M$, the complexity may be described by the function $f(n)$ which gives the running tine and/or storage space requirement of the algorithm in terms of the size $n$ of the input data. In most of the cases, the storage space required by an algorithm is simply a multiple of the data size. Accordingly, unless otherwise stated or implied, the term complexity shall refer to the running time of the algorithm. The complexity function $f(n)$, which we assume gives the running time of an algorithm, usually depends not only on the size $n$ of the input data but also on the particular data. In the complexity theory, the following two cases are usually investigated:

1. Worst Case : The maximum value of $f(n)$ for any possible input.
2. Average Case : The expected value of $f(n)$.

The analysis of average case assumes a use of probability distribution for the input data and we assume that the possible permutations of a data set are equally likely. Also, the following result is used for an average case:

Suppose the numbers $n_1, n_2, ........, n_k$ occur with respective probabilities $p_1, p_2, ........, p_k$, then the expectation or average value $E$ is given by

$E = n_1 p_1 + n_2 p_2 + ............. + n_k p_k$.

### 1.4.6.1 Standard Functions Measuring Complexity of Algorithms

Algorithms are generally compared or analysed on the basis of their complexity which is further measured in terms of the size of input data $n$ described by the mathematical function $f(n)$. As $n$ grows, complexity of algorithm $M$ also increases and our interest is to measure this rate of growth. For the purpose, we compare $f(n)$ with some standard functions with different rate growths such that $\log_2 n, n, n\log_2 n, n^2, n^3, 2^n$. Now, complexity of any algorithm is measured in terms of these standard functions and we use a special notation for this, called Big-O notation, as defined below:

**Big-** O **:** Let $f(x)$ and $g(x)$ are functions defined on the set (or subset) of real numbers. Then, $f(x)$ is called order of $g(x)$ or big- O of $g(x)$, written as $f(x) = O(g(x))$, if there exist a real number $m$ and a positive constant $c$ such that for all $x \geq m$, we have $|f(x)| \leq c|g(x)|$.

To show that $f(x) = O(g(x))$ we have to find the value of $m$ and $c$. Further, big-O gives an upper bound on number of key operations or we can say that big-O gives information about maximum number of key operations. For getting lower bound, we define the function big-omega ($\Omega$).

**Big-omega :** Let $f(x)$ and $g(x)$ are functions defined on the set (or subset) of real numbers. Then, $f(x) = \Omega(g(x))$ of there exist positive constants $c$ and $k$ such that $|f(x)| \geq c|g(x)|$ for all $x \geq k$. Further, $f(x)$ is called big-omega of $g(x)$.

**Big-theta :** Let $f(x)$ and $g(x)$ are functions defined on the set (or subset) of real numbers. Then, $f(x) = \Theta(g(x))$ of there exist positive constants $c_1, c_2$ and $k$ such that $c_1|g(x)| \leq |f(x)| \leq c_2|g(x)|$ for all $x \geq k$. Further, $f(x)$ is called big-theta of $g(x)$ if $f(x)$ is both big-O and big-omega of $g(x)$.

### 1.4.7 Growth Rate Functions

The time efficiency of almost all the algorithms can be characterized by the following growth rate functions:

1. **O($l$)-Constant Time :** This means that the algorithm requires the same fixed number of steps regardless of the size of the task.

   For Example (Assuming a reasonable implementation of the task) :

i. Push and pop operations for a stack (containing n elements);
ii. Insert and remove operations for a queue.

2. **O($n$)-Linear Time :** This means that the algorithm requires a number of steps proportional to the size of the task.
   For Example (Assuming a reasonable implementation of the task) :
   i. Traversal of a tree with $n$ nodes;
   ii. Calculating $n$-factorial or $n^{th}$ Fibonacci number by using the method of iteration.
3. **O($n^2$)-Quadratic Time :** This means that the algorithm requires a number of steps proportional to the square of size of the task.
   For Example :
   i. Comparing two dimensional array of size $n$ by $n$;
   ii. Find duplicates in an unsorted list of $n$ elements (implemented with two nested loops).
4. **O($\log n$)-Logarithmic Time :**
   For Example :
   i. Binary search in a sorted list of $n$ elements;
   ii. Insert and find operations for a binary search tree with $n$ nodes.
5. **O($n \log n$)-"$n \log n$" Time :**
   For Example :
   i. More advanced sorting algorithms - quicksort, mergesort.
6. **O($a^n$)($a > 1$) -Exponential Time :**
   For Example :
   i. Recursive Fibonacci Implementation;
   ii. Generating all permutations of $n$ symbols.

**Remarks : (i)** The order of asymptotic behavior of the above described functions is

O($l$)<O($\log n$)<O($n$)<O($n \log n$)<O($n^2$)-O($n^3$)<O($a^n$)

So, the best time is the constant time and the worst time is the exponential time and polynomial growth is considered manageable as compared to exponential growth.

**(ii)** If a function (which describes the order of growth of an algorithm) is a sum of several terms, its order of growth is determined by the **fastest growing term**. In particular, if we have a polynomial of the form

$$p(n) = a_k n^k + a_{k-1} n^{k-1} + .................... + a_1 n + a_0 ,$$

then its growth is of the order $n^k$ i.e., $p(n) = O(n^k)$.

### 1.4.8 Some Important Examples

**Example 1 :** Find $\lfloor \log_2 100 \rfloor$

**Sol.** $\because 2^6 = 64$ and $2^7 = 128$.

so $6 < \log_2 100 < 7$

$\Rightarrow \log_2 100 = 6.$

**Example 2 :** Show that $f(x) = x^2 + 2x + 1$ is O($x^2$).

**Sol.** Let $x \geq 1$

which gives $1 \leq x \leq x^2$ ....(1)

Now, $|f(x)| = |x^2 + 2x + 1| \leq |x^2| + |2x| + |1|$ [since $|x + y| \leq |x| + |y|$ ]

$\Rightarrow |f(x)| = x^2 + 2x + 1 \leq x^2 + 2x^2 + x^2 = 4x^2$ [using (1)]

$\Rightarrow |f(x)| \leq 4|x^2| \forall x \geq 1$

which gives $f(x) =$ O($x^2$)

**Example 3 :** Suppose the polynomial $P(x) = a_0 + a_1 x + a_2 x^2 + .........+ a_n x^n$ is of degree $n$. Show that $P(x) =$ O($x^n$).

**Sol.** Let $x \geq 1$

so, $1 \leq x \leq x^2 \leq x^3 \leq .................. \leq x^n$ ....(1)

Now, $|P(x)| = |a_0 + a_1 x + a_2 x^2 + .........+ a_n x^n|$

$\leq |a_0| + |a_1 x| + |a_2 x^2| + .........+ |a_n x^n|$ [since $|x + y| \leq |x| + |y|$ ]

$= |a_0| + |a_1| x + |a_2| x^2 + .........+ |a_n| x^n$

$= |a_0|.1 + |a_1| x + |a_2| x^2 + .........+ |a_n| x^n$

$\leq |a_0| x^n + |a_1| x^n + |a_2| x^n + .........+ |a_n| x^n$ [using (1)]

$= [|a_0| + |a_1| + |a_2| + .........+ |a_n|] x^n = cn^m$

where $c = |a_0| + |a_1| + |a_2| + .........+ |a_n|$

so, $|P(x)| \leq c|x^n| \forall x \geq 1$

which gives $P(x) =$ O($x^n$) .

**Example 4 :** Find Big- O notation for $\log \angle n$. Further give Big- O estimate for $f(n) = 3n \log \angle n + (n^2 + 3) \log n$

**Sol.** Let $n$ be any natural number.

As we know $\angle n = 1.2.3............n$

Now, $1 < 2 < 3 < 4...................... \leq n$

so, $|\angle n| = 1.2.3.................n \le n.n.n...............n = n^n$

$\Rightarrow |\angle n| \le 1.n^n$ for all $n \ge 1$

so, $\angle n = O(n^n)$ with $c = 1, m = 1$

$\Rightarrow \log \angle n = O(\log n^n) = O(n \log n)$ ...(1)

For the second part, Let $n \ge 1$

so, $1 \le n \le n^2$ ...(2)

so, $|f(n)| = |3n \log \angle n + (n^2 + 3)\log n|$

$\le |3n \log \angle n| + |(n^2 + 3)\log n|$ [since $|x + y| \le |x| + |y|$ ]

$\le 3n.n \log n + (n^2 + 3.n^2)\log n$ [using (1) and (2)]

$= 3n^2 \log n + 4n^2 \log n$

$\Rightarrow |f(n)| \le 7.|n^2 \log n|$ for all $n \ge 1$

so, f(n) is O($n^2 \log n$) with $c = 7, m = 1$.

**Example 5 :** Prove that $f(x) = 8x^3 + 5x^2 + 7$ is $\Omega(g(x))$ where $g(x) = x^3$.

**Sol.** Let $x \ge 0$

then, $x^2 \ge 0$

Now, $|f(x)| = |8x^3 + 5x^2 + 7| \ge 8x^3$ $[\because 5x^2 + 7 \ge 0]$

$\Rightarrow |f(x)| \ge 8.|x^3|$ $\forall x \ge 0$

$\Rightarrow |f(x)| \ge 8.|g(x)|$ $\forall x \ge 0$

Hence $f(x)$ is $\Omega(g(x))$ where $c = 8, k = 0$.

### 1.4.9 Summary

In this lesson, we have studied about the algorithms. From our study, we can say that an algorithm is a sequence of instructions. Each individual instruction must be carried out, in its proper place, by the person or machine for whom the algorithm is intended. Consequently, an algorithm should always be considered in the context of certain assumptions. In more detail, we have discussed about the efficiency and complexity of algorithms, on the basis of which, we have learnt the procedure to compare the algorithms. Further, we have also given an idea of recursive algorithms that may be useful for understanding the recurrence relations which will be discussed in the next part of this unit.

### 1.4.10 Key Concepts

Algorithm, Algorithm Efficiency, Recursive Algorithm, Complexity of Algorithms, Big-O, Big-Omega, Big-Theta, Growth rate functions.

### 1.4.11 Long Questions

1. Discuss various aspects of algorithm efficiency.
2. Discuss in detail about the growth rate functions.

### 1.4.12 Short Questions

1) Show that $7x^2 - 9x + 4 = O(x^2)$.
2) Let $U + \{a, b, c, ........, x, y, z\}$ and $A = \{a, e, i, o, u\}$ . Find the characteristic function of $A$ .
3) Show that $g(n) = n^2(7n - 2)$ is $O(n^3)$.
4) Show that $x^4 + 9x^3 + 4x + 7$ is $O(x^4)$.
5) Find Big-$O$ notation for $\angle n$.

### 1.4.13 Suggested Readings

1. Norman L. Biggs, *Discrete Mathematics*, Oxford University Press.
2. Harmohan Sharma, Ganesh Kumar Sethi, *Discrete Mathematics,* Sharma Publications, Jalandhar.
3. C.L. Liu, Elements of Discrete Mathematics (Second Edition), McGraw Hill, International Edition, Computer Science Series, 1986.

**LESSON NO. 1.5**

**Author : Dr. Chanchal**

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# THE BASICS OF COUNTING, PIGEONHOLE PRINCIPLE & MATHEMATICAL INDUCTION

**Structure :**



## 1.5.1 Objectives

In this lesson, our prime objectives are:

- To study fundamental principle of counting
- To discuss about arrangement of n things out of r things and choosing r things out of n things i.e. permutation and combination
- To study basics of probability theory
- To study Pigeonhole principle and its extended from with the help of examples
- To study Principle of Mathematical Induction

## 1.5.2 Introduction

Firstly, we defined factorial n as :

**Def :** The product of all positive integers from 1 to n is called factorial n. It is denoted by $\lfloor n$ or n !.

**For Example :** $\lfloor 4 = 4.3.2.1 = 24$

**Note :** (i) $\lfloor n = n(n-1)(n-2)(n-3).....3.2.1 = n\lfloor n-1$

(ii) $\lfloor 0 = 1, \lfloor 1 = 1$

(iii) Factorial of proper fraction and negative integer is not defined.

**Example 1 :** Prove that $\dfrac{\lfloor 2n}{\lfloor n} = 1.3.5........(2n-1).2^n$

**Sol.** $$\text{L.H.S.} = \frac{\lfloor 2n}{\lfloor n} = \frac{1.2.3.4.5.6......(2n-1)(2n)}{\lfloor n}$$

$$= \frac{[1.3.5..........(2n-1)][2.4.6.........(2n)]}{\lfloor n}$$

$$= \frac{[1.3.5.......(2n-1)]\ .2^n[1.2.3.........n]}{\lfloor n}$$

$$= \frac{[1.3.5........(2n-1)].2^n\lfloor n}{\lfloor n} = 1.3.5\ .........(2n-1)\,2^n = \text{R.H.S.}$$

### 1.5.3 Fundamental Principle of Counting

If one operation can be performed in 'm' different ways and if corresponding to each of these m ways of performing the first operation, there are 'n' different ways of performing the second operation, then the number of different ways of performing the two operations taken together is m × n.

**Example 2 :** How many numbers can be formed from the digits 1, 2, 3, 9 if repetition of digits is not allowed ?

**Sol. (a) Numbers with one digit :** There are four digits, hence four numbers of one digit can be formed with the help of these digits.

Hence, number of one digit numbers = 4.

**(b) Numbers with two digits :** First place of two digit number can be filled in 4 ways and the second place can be filled in 3 ways.

Hence, number of two digit numbers = 4 × 3 = 12.

**(c) Numbers with three digits :**

Number of three digits number = 4 × 3 × 2 = 24.

**(d) Number with four digits :**

Number of four digits numbers = 4 × 3 × 2 × 1 = 24.

Hence, total number of digits formed with the given digits

= 4 + 12 + 24 + 24 = 64.

**Example 3 :** Given 5 flags of different colours, how many different signals can be generated if each signal requires the use of 2 flags, one below the other ?

**Sol.** Number of flags = 5

Number of flags required for a signal = 2

First place of signal can be filled in 5 ways and corresponding to each way of filling the first place, there are four ways of filling the second place.

$\therefore$ by fundamental principle of counting.

total number of signals generated = 5 × 4 = 20.

## 1.5.4 Permutation

**Def :** It is an arrangement that can be made by taking some or all of a number of given things. It is denoted by $^nP_r$ which means number of permutations of n different things taken 'r' at a time. Further, $^nP_r = \frac{\lfloor n}{\lfloor n-r}$.

**Illustration :** Consider three letters a, b, c. Now, the permutations of three letters taken two at a time are : ab, bc, ca, ba, cb, ac. Which are 6 in number.

Mathematically, $^nP_r = {}^3P_2 \frac{\lfloor 3}{\lfloor 3-2} = \frac{\lfloor 3}{\lfloor 1} = 6$

**Note :** $^nP_r$ is also writen as P (n, r)

**Example 4 :** Find n if P (2n, 3) = 100 P (n, 2).

**Sol.** Since P (2n, 3) = 100 P (n, 2)

$\therefore$ $^{2n}P_3 = 100\ {}^nP_2 \Rightarrow (2n)(2n-1)(2n-2) = 100\ n(n-1)$

$\Rightarrow$ $4n(n-1)(2n-1) = 100n(n-1) \Rightarrow 2n-1 = 25$

$\Rightarrow$ $n = 13$

## 1.5.5 Practical Problems Involving Permutation

**Example 5 :** How many words, with or without meaning, can be formed using all the letters of the word EQUATION, using each letter exactly once. Further, how many words can be formed if each word is to start with a vowel.

**Sol.** Number of letters is EQUATION = 8

No. of letters to be taken at a time = 8

$\therefore$ required no. of words $= {}^8P_8 = \frac{\underline{|8}}{\underline{|0}} = 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 40320$

Fruther, vowels in EQUATION are E,U, A, I, O i.e. 5 vowels. If the first place is to be filled with vowel, it can be filled in 5 ways. Now, remaining 7 places can be filled up with 7 letters in $\underline{|7}$ or ${}^7P_7$ ways.

$\therefore$ required no. of words = $5 \times \underline{|7} = 5 \times (7\times6\times5\times4\times3\times2\times1) = 25200$

**Example 6 :** In how many ways can 5 books on Chemistry and 4 books on Physics be arranged on a shelf so that the books on same subject remain together ?

**Sol.** Consider the 5 books on Chemistry as one book and also 4 books on Physics as one book.

$\therefore$ two books can be arranged in $\underline{|2}$ ways

Also the 5 books on Chemistry can be arranged among themselves in $\underline{|5}$ ways and 4 book on Physics in themselves in $\underline{|4}$ ways.

$\therefore$ required number of ways = $\underline{|2} \times \underline{|5} \times \underline{|4}$

$= (2 \times 1) \times (5 \times 4 \times 3 \times 2 \times 1) \times (4 \times 3 \times 2 \times 1)$

$= 2 \times 120 \times 24 = 5760.$

**Example 7 :** In how many ways can 4 boys and 3 girls be seated in a row so that no two girls are together ?

**Sol.** Let 4 boys the $B_1$, $B_2$, $B_3$, $B_4$

$\times\ B_1 \times B_2 \times B_3 \times B_4 \times$

$\because$ no two girls are together

$\therefore$ three girls can be arranged in 5 '×' marked places in ${}^5P_3$ ways.

Also 4 boys can be arranged among themselves $\underline{|4}$ ways

$\therefore$ required number of ways = ${}^5P_3 \times \underline{|4}$

$= (5 \times 4 \times 3) \times (4 \times 3 \times 2 \times 1)$

$= 60 \times 24 = 1440$

**Example 8 :** How many numbers greater than 40000 can be formed using the digits 1, 2, 3, 4 and 5 if each digit is used only once in each number ?

**Sol.** Given digits are 1, 2, 3, 4, 5

$\therefore$ number of given digits = 5

Number of digits to be taken at a time = 5

Since number is to be greater than 4000

$\therefore$ first digit from left should be either 4 or 5

i.e. first place can be filled in 2 ways.

Remaining 4 places with 4 digits can be filled in $^4P_4$ ways

$\therefore$ required numbers = $2 \times {}^4P_4 = 2 \times (4 \times 3 \times 2 \times 1) = 2 \times 24 = 48$.

**Example 9 :** How many different signals can be formed with five given flags of different colours ?

**Sol.** Number of flags = 5

A signal may formed by hoisting any number of flags at a time.

Number of signals by hoisting one flags at a time = $^5P_1$

Number of signals by hoisting two flags at a time = $^5P_2$

Number of signals by hoisting three flags at a time = $^5P_3$

Number of signals by hoisting four flags at a time = $^5P_4$

Number of signals by hoisting five flags at a time = $^5P_5$

$\therefore$ total number of signals formed

$= {}^5P_1 + {}^5P_2 + {}^5P_3 + {}^5P_4 + {}^5P_5$

$= 5 + 20 + 60 + 120 + 120 = 325.$

**Result :** The number of permutations of n things taken all at a time when p of them are alike and of one kind, q of them are alike and of second kind, all other being different is given by $\dfrac{\underline{|n}}{\underline{|p} \times \underline{|q}}$.

**Example 10 :** How many permutations of the letter of word APPLE are there ?

**Sol.** No. of given letters = 5

No. of P's = 2

$\therefore$ required no. of permutations $= \dfrac{\underline{|5}}{\underline{|2}} = \dfrac{5 \times 4 \times 3 \times 2 \times 1}{2 \times 1} = 60$

**Example 11 :** How many numbers greater than 1000000 can be formed by using the digits 1, 2, 0, 2, 4, 2, 4.

**Sol.** Given digits are 1, 2,0, 2, 4, 2, 4

$\therefore$ total number of digits = 7

with Number of 2's = 3 and Number of 4's = 2

Number of digits to be taken at a time = 7

$\therefore$ numbers formed = $\dfrac{\underline{|7}}{\underline{|3} \times \underline{|2}} = \dfrac{7 \times 6 \times 5 \times 4 \times \underline{|3}}{\underline{|3} \times (1 \times 2)} = 420$

These numbers also include those numbers which have 0 at the extreme left position.

Numbers having 0 at the extreme left position = $\frac{\lfloor 6}{\lfloor 3 \times \lfloor 2} = \frac{6 \times 5 \times 4 \times \lfloor 3}{\lfloor 3 \times (1 \times 2)} = 60$

$\therefore$ required number of numbers = 420 – 60 = 360.

**1.5.5.1 Circular Permutations**

FInd the number of ways in which n persons can be arranged at a round table.

**Proof :** When n persons are sitting around a circular table, then there is no first and last person. Let us fix the position of one person. The remaining (n–1) persons can how be arranged in the remaining (n–1) places in ${}^{n-1}P_{n-1}$ i.e., $\lfloor n-1$ ways.

$\therefore$ required number of ways = $\lfloor n-1$ .

**Clockwise and Anti-clockwise Permutations**

The total number of circular permutations can be divided into two types :

(i) Clockwise (ii) Anti-clockwise.

In two such arrangements each person has the same neighbour though in the reverse order and either of these arrangements can be obtained from the other by just over-turning the circle. If in this case, no distinction is made between clockwise and anti-clockwise arrangements then the two such arrangements are considered as only one distinct arrangement.

Hence the number of circular permutations in such cases $= \frac{1}{2} \lfloor n-1$

**Note.** Questions on necklaces with beads if different colours are to be tackled by the above formula, as in this case also there is no difference between clockwise and antiwise arrangements.

**Example 12 :** In how many ways can 8 girls be seated at a round table provided Parveen and Vipul are not to sit together ?

**Sol.** Total number of girls = 8.

$\therefore$ number of ways in which they can be arranged on a round table

$$= \lfloor 8-1 = \lfloor 7 = 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 5040$$

Consider two girls Parveen and Vipul as one. Therefore 7girls can be arranged in $\lfloor 7-1 = \lfloor 6$ ways. Also the two girls can be arranged among themselves in $\lfloor 2$ ways.

$\therefore$ number of arrangements in which two particular girls are always together

$$= \lfloor 6 \times \lfloor 2 = (6 \times 5 \times 4 \times 3 \times 2 \times 1) \times (2 \times 1) = 1440$$

$\therefore$ required number of arrangements = 5040 – 1440 = 3600.

## 1.5.6 Combination

**Def :** It is a group (or selection) that can be made by taking some or all of a number of given things at a time. It is denoted by $^nC_r$ which means number of combinations of n different things taken 'r' at a time. Further, $^nC_r = \frac{\lfloor n}{\lfloor r \lfloor n-r}$.

**Illustration :** Consider three letters a,b, c. The groups of there 3 letters taken two at a time are ab, bc, ca. As far as group is concerned ac or ca is the same group because in a group, we are concerned with the number of things contained unlike with the case of arrangement where we have to consider the order of things also.

**Note :** (i) $^nC_r = {^nC_{n-r}}$

(ii) $^nC_r + {^nC_{r-1}} = {^{n+1}C_r}$

(iii) The total number of combinations of n different things by taking some or all at a time i.e. $^nC_1 + {^nC_2} + {^nC_3} + \ldots\ldots\ldots + {^nC_n}$ is given by $2^n-1$.

The above results can be proved very easily and left as an exercise for the reader.

## 1.5.7 Practical Problems Involving Combinations

**Example 13 :** A mathematics paper consists of 10 questions divided into two parts I and II. Each part containing 5 questions. A student is required to attempt 6 questions in all, talking at least 2 questions from each part. In how many ways can the student select the questions ?

**Sol.** Number of questions in part I = 5

Number of question in part II = 5

| Part I | Part II |
|---|---|
| 5 | 5 |
| 2 | 4 |
| 3 | 3 |
| 4 | 2 |

Number of questions to be attempted = 6

$\because$ each selection contains at least 2 from each part

$\therefore$ different possibilities are

(i) 2 from part 1, 4 from part II

(ii) 3 from part 1, 3 from part II

(iii) 4 from part 1, 2 part II

$\therefore$ required number of ways

$$= {}^5C_2 \times {}^5C_4 + {}^5C_3 \times {}^5C_3 + {}^5C_4 \times {}^5C_2$$

$$= {}^5C_2 \times {}^5C_1 + {}^5C_2 \times {}^5C_2 + {}^5C_1 \times {}^5C_2$$

$$= \frac{5\times4}{1\times2} \times \frac{5}{1} + \frac{5\times4}{1\times2} \times \frac{5\times4}{1\times2} + \frac{5}{1} \times \frac{5\times4}{1\times2}$$

$$= 10\times5 + 10 \times 10 + 5 \times 10 = 50 + 100 + 50 = 200.$$

**Example 14 :** A committee of 5 is to be selected from among 6 boys and 5 girls. Determine the number of ways of selecting the committee if it is to consist of all least 1 boy and 1 girl.

**Sol.** Number of boys = 6, Number of girls = 5

| Boys | Girls |
|---|---|
| 6 | 5 |
| 1 | 4 |
| 2 | 3 |
| 3 | 2 |
| 4 | 1 |

Committee is to be formed of 5.

$\because$ committee is to include at least 1 boy and 1 girl

$\therefore$ different possibilities are

(i) 1 boys, 4 girls (ii) 2 boys, 3 girls

(iii) 3 boys, 2 girls (iv) 4 boys, 1 girl

$\therefore$ required number of ways

$$= {}^6C_1 \times {}^5C_4 + {}^6C_2 \times {}^5C_3 + {}^6C_3 \times {}^5C_2 + {}^6C_4 \times {}^5C_1$$

$$= {}^6C_1 \times {}^5C_1 + {}^6C_2 \times {}^5C_2 + {}^6C_3 \times {}^5C_2 + {}^6C_2 \times {}^5C_1$$

$$= \frac{6}{1} \times \frac{5}{1} + \frac{6\times5}{1\times2} \times \frac{5\times4}{1\times2} + \frac{6\times5\times4}{1\times2\times3} \times \frac{5\times4}{1\times2} + \frac{6\times5}{1\times2} \times \frac{5}{1}$$

$$= 30 + 150 + 200 + 75 = 455.$$

**Example 15 :** The number of diagonals of a polygon is 20. Find the number of its sides.

**Sol.** Let number of sides of polygon = n, $\therefore$ Number of points = n

Number of lines formed $= {}^nC_2 = \dfrac{n\,(n-1)}{2}$

$\therefore$ number of diagonals $= \dfrac{n\,(n-1)}{2} - n$

From given condition, $\frac{n(n-1)}{2} - n = 20$

$\therefore$ $n^2 - n - 2n = 40$ $\Rightarrow$ $n^2 - 3n - 40 = 0$

$\Rightarrow$ $(n-8)(n+5) = 0$ $\Rightarrow$ $n = 8, -5$

Rejecting n = –5 as number of sides cannot be negative, we get, n = 8

$\therefore$ number of sides = 8.

**Example .16 :** Ram has 5 friends. In how many ways can be invite one or one of them o a party ?

**Sol.** Number of friends = 5

Ram can invite one friend, two friends, three friends, four friends, four friends or five friends.

$\therefore$ required number of ways $= {}^5C_1 + {}^5C_2 + {}^5C_3 + {}^5C_4 + {}^5C_5$

$= {}^5C_1 + {}^5C_2 + {}^5C_2 + {}^5C_1 + {}^5C_0$

$= \frac{5}{1} + \frac{5 \times 4}{1 \times 2} + \frac{5 \times 4}{1 \times 2} + \frac{5}{1} + 1$

$= 5 + 10 + 10 + 5 + 1 = 31.$

### 1.5.8 Pigeonhole Principle

**Simple Form :** If n pigeons are assigned to m pigeonholes and m < n, then there is at least one pigeonhole that contains two or more pigeons.

**Proof :** Label n pigeons with the numbers 1 to n and m pigeonholes with the numbers 1 to m. Starting with pigeon 1 and pigeonhole 1, assign each pigeon in order to the pigeonhole with the same number. So we can assign as many pigeons as possible to distinct pigeonholes. Since the number m of pigeonholes is less than the number n of the pigeons, so n – m pigeons are left that are not assigned to a pigeonhole. Therefore, there is atleast one pigeonhole that will be assigned one or more than one pigeon again.

$\therefore$ there is at least one pigeonhole that contains two or more pigeons.

**Extended Form :** If n pigeons are assigned m pigeonholes, where n is sufficiently large as compared to m, then one of the pigeonholes must contain at least $\left[\frac{n-1}{m}\right] + 1$ pigeons.

**Proof :** Assume that the result is false

$\therefore$ each pigeonhole does not contain more than $\left[\frac{n-1}{m}\right]$ pigeon.

$\therefore$ maximum possible number of pigeons $= \left[\frac{n-1}{m}\right] m \le \frac{n-1}{m} . m$

$= n - 1$

This contradicts the given result that number of pigeons is n.

$\therefore$ our supposition is wrong.

Hence the result.

**Example 17 :** Use Pigeonhole Prnciple to show that if sven numbers from 1 to 12 are chosen, then two of them will add upto 13.

**Sol.** The sets which add upto 13 are

{1, 12}, {2, 11}, {3, 10}, {4, 9}, {5, 8}, {6, 7}.

By Pigeonhole principle, if we have to choose seven numbers then we must take at least two numbers belonging to one set. Thus two of the seven numbers will definitely add upto 13.

**Example 18 :** Use Pigeonhole Prinicple to prove that an injection cannot exist between a finite set A and a finite set B if Cardinality of A is greater than Cardinality of B.

**Sol.** Let n (A) = a n(B) = b where a > b.

Consider elements of Set B as pigeonholes and elements of set A as pigeons. As no. of pigeons are more than pigeon holes so at least two pigeons will have same pigeonholes or we can say $\exists$ x, y $\varepsilon$ A such that f (x) = f(y). But x $\neq$ y.

Sof : A $\rightarrow$ B is not injective.

**Example 19 :** How many people among 200000 people are born at same time (hour, minute, seconds) ? Use Pigeonhole principle to find it.

**Sol.** Total number of persons = 200000

Total number of seconds in a day = 24 × 60 × 60 = 86,400

Here, we have to assign a time to each person

So person are like pigeons time is like pigeonhole.

Number of pigeons (n) = 200000

Number of pigenholes (m) = 86,400

Min. number of persons having same birthday

$$= \left[\frac{n-1}{m}\right] + 1 = \left[\frac{200000-1}{86400}\right] + 1 = \left[\frac{1,99,999}{86400}\right] + 1 = 2 + 1 = 3 .$$

## 1.5.9 Mathematical Induction

**Principle of Mathematical Induction :**

Let P(n) be the given statement. Then to prove the validity of P(n) we have to perform following three steps :

**(i)** **Basis :** First we prove the given statement is true for n = 1 i.e. P(1) is true.

**(ii)** **Assumption :** We assume result is true for n = k.

**(iii)** **Induction :** We prove that the given statement is true for n = k + 1 i.e. P(k + 1) is true.

Then we conclude by principle of mathematical induction that statement is true for all $n \in N$.

**Example 20 :** Use Mathematical Induction to show that $1 + 2 + 4 + \ldots\ldots 2^n = 2^{n+2} - 1$

**Sol.** Let P (n) be the statement

$$1 + 2 + 4 \ldots\ldots\ldots 2^n = 2^{n+1} - 1 \text{ or } 2^0 + 2^1 + 2^2 \ldots\ldots 2^n = 2^{n+1} - 1$$

**Basis :** First we show that P(1) is true, so put n = 1

L.H.S R.H.S

$2^0 + 2^1 = 1 + 2 + 3$ $\quad 2^{14} - 1 = 4 - 1 = 3$

3 = 3 :P (1) is true

**Assumption :** Suppose that P(k) is true, so taking n = k

$$2^0 + 2^1 + 2^2 \ldots\ldots\ldots 2^k = 2^{k+1} - 1$$

**Induction :** Now we prove P (k + 1) is true, taking n = k + 1

$$2^0 + 2^1 + 2^2 \ldots\ldots\ldots 2^k + 2^{k+1} = 2^{k+1+1} - 1$$

L.H.S. $2^0 + 2^1 + 2^2 \ldots. 2^k + 2^{k+1} = (2^0 + 2^1 + 2^2 \ldots\ldots 2^k) + 2^{k+1}$

$$= 2^{k+1} - 1 + 2^{k+1}$$

$$= 2^{k+1} + 2^{k+1} - 1 = 2.2^{k+1} - 1 = 2^{k+1+1} - 1 = \text{R.H.S}$$

$\therefore$ P (k + 1) is true.

Hence P(n) is true by Induction.

**Example 21 :** Prove by induction $a + ar + ar^2 \ldots\ldots. ar^{n-1} = \dfrac{a(1-r^n)}{1-r}, r \neq 0$

**Sol.** Basis : for n = 1

L.H.S R.H.S

$a \cdot r^{1-1}$ $\quad \dfrac{a(1-r^1)}{1-r}, a$

a = a

$\therefore$ Result is true for n = 1

**Assumption :** Let result is true for n = k

$$a + ar + ar^2 \ldots\ldots. + ar^k = a \; \frac{(1-r^k)}{1-r}, r \neq 0 \qquad \ldots (1)$$

**Induction :** Put n = k + 1

$$a + ar + ar^2 \dots\dots\dots + ar^k = a\ \frac{(1-r^{k+1})}{1-r}$$

$$\text{L.H.S.} = a + ar + ar^2 \dots\dots\dots + ar^k = a + ar + ar^2 \dots\dots\dots ar^{k-1} + ar^k$$

$$= a\ \frac{(1-r^k)}{1-r} + ar^k = a\left[\frac{1-r^k+r^k-r^{k+1}}{1-r}\right] = a\left[\frac{1-r^{k+1}}{1-r}\right] = \text{R.H.S.}$$

So, result is true for n = k + 1.

**Example 22 :** Prove by induction that 21 divides $4^{n+1} + 5^{2n-1}$.

**Sol. Basis :** For n = 1

$4^{1+1} + 5^{2.1-1} = 4^2 + 5^1 = 16 + 5 = 21$ which divides 21

$\therefore$ P(1) is true.

**Assumption :** Let result is true for n = k.

i.e. 21 divides $4^{k+1} + 5^{2k-1} \Rightarrow 4^{k+1} + 5^{2k-1} = 21m$

$\Rightarrow$ $5^{2k-1} = 21\ m - 4^{k+1}$ ... (1)

**Induction :** Put n = k + 1

$4^{k+1+1} + 5^{2\ (k+1)-1} = 4^{k+1} .\ 4^1 + 5^{2k-1} .\ 5^2$

$= 4^{k+1} .\ 4 + (21m - 4^{k+1})\ 25$ [Using (1)]

$= 4^{k+1} .\ 4 + 21m\ .\ 25 - 4^{k+1}.25 = 4^{k+1}\ (4 - 25) + 21m\ .\ 25 = 4^{k+1}\ (-21) + 21m.25$

$= 21\ (-4^{k+1} + 25m)$ which is divisible by 21.

### 1.5.10 Summary

In this lesson, we have studied permutations including circular permutations and combinations with solutions to some of their practical problems. We have discussed about the probability and defined many important terms related to it. With the help of this lessons, students have also attained the knowledge of Pigeonhole Principle alongwithits extended form and Principle of Mathematical Induction. All the concepts are made easily understandable with the help of simple practical examples.

### 1.5.11 Key Concepts

Fundamental principle of counting, Factorial, Permutation, Combination, Circular permutation, Random experiment, Event, Elementary events, Sample space, Sure event, Impossible event, Equally likely events, Mutually excusive events, Probability, Addition theorem, Multiplication theorem, Independent events, Pigeonhole principle, Principle of Mathematical Induction.

### 1.5.12 Long Questions

1. Prove that $C\,(2n, n) = \dfrac{2^n\,[1.3.5\dots\dots\dots(2n-1)]}{\underline{|n}}$

2. A group consists of 4 girls and 7 boys. In how many ways can a team of 5 numbers be selected if the team has (i) no girls ? (ii) atleast one boy and one girl ?

3. There are 15 points in a plane 1 no three of which are in the same straight line excepting 4, which are collinear. Find the no. of (i) straight lines (ii) triangles, formed by joining them.
4. A sport team of 11 students is to be constituted, choosing atleast 5 from class XI and atleast 5 from class XII. If there are 20 students in each of these classes, in how many ways can the team be constituted.

**1.5.13 Short Questions**

1. In how many different ways, the letters of the word ALGEBRA can be arranged in a row if two A's are never together ?
2. Find the number of different 8 letter words formed from the letters of word TRIANGLE if each word is to have both consonants and vowels together.
3. In how many ways 4 boys and 4 girls be seated at a round table provided each boy is to be between two girls ?
4. How many people must you have to guarantee that atleast 12 of them will have birthdays on the same day of the week ? Use pigeonhole principle.
5. Use pigeonhole principle to show that if seven numbers from 1 to 12 are chosen, then two of them will add upto 13.

**1.5.14 Suggested Readings**

1. Norman L. Biggs, *Discrete Mathematics*, Oxford University Press.
2. Harmohan Sharma, Ganesh Kumar Sethi, *Discrete Mathematics,* Sharma Publications, Jalandhar.
3. C.L. Liu, Elements of Discrete Mathematics (Second Edition), McGraw Hill, International Edition, Computer Science Series, 1986.

# Mandatory Student Feedback Form

https://forms.gle/KS5CLhvpwrpgjwN98

Note: Students, kindly click this google form link, and fill this feedback form once.